THE BOLZANO-WEIERSTRASS THEOREM IS THE JUMP OF WEAK KŐNIG'S LEMMA

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Abstract. We classify the computational content of the Bolzano-Weierstraß Theorem and variants thereof in the Weihrauch lattice. For this purpose we first introduce the concept of a derivative or jump in this lattice and we show that it has some properties similar to the Turing jump. Using this concept we prove that the derivative of closed choice of a computable metric space is the cluster point problem of that space. By specialization to sequences with a relatively compact range we obtain a characterization of the Bolzano-Weierstraß Theorem as the derivative of compact choice. In particular, this shows that the Bolzano-Weierstraß Theorem on real numbers is the jump of Weak Kőnig's Lemma. Likewise, the Bolzano-Weierstraß Theorem on the binary space is the jump of the lesser limited principle of omniscience LLPO and the Bolzano-Weierstraß Theorem on natural numbers can be characterized as the jump of the idempotent closure of LLPO (which is the jump of the finite parallelization of LLPO). We also introduce the compositional product of two Weihrauch degrees f and g as the supremum of the composition of any two functions below f and g, respectively. Using this concept we can express the main result such that the Bolzano-Weierstraß Theorem is the compositional product of Weak Kőnig's Lemma and the Monotone Convergence Theorem. We also study the class of weakly limit computable functions, which are functions that can be obtained by composition of weakly computable functions with limit computable functions. We prove that the Bolzano-Weierstraß Theorem on real numbers is complete for this class. Likewise, the unique cluster point problem on real numbers is complete for the class of functions that are limit computable with finitely many mind changes. We also prove that the Bolzano-Weierstraß Theorem on real numbers and, more generally, the unbounded cluster point problem on real numbers is uniformly low limit computable. Finally, we also provide some separation techniques that allow to prove non-reducibilities between certain variants of the Bolzano-Weierstraß Theorem.

1. Introduction

In this paper we continue the programme to classify the computational content of mathematical theorems in the Weihrauch lattice. This programme has been started recently in [GM09, BG11b, BG11a, Pau10b, BdBP, Pau10a] and the basic idea is to interpret statements of the form

$$(\forall x \in X)(x \in D \Longrightarrow (\exists y \in Y)(x, y) \in A)$$

as partial multi-valued functions $f :\subseteq X \rightrightarrows Y, x \mapsto \{y \in Y : (x, y) \in A\}$ with $\mathrm{dom}(f) = D$. Here the symbol "$\subseteq$" is used to indicate that the function is partial and "$\rightrightarrows$" denotes that it is multi-valued. The translation of theorems into such multi-valued functions is straightforward and these functions are directly the elements of the Weihrauch lattice. The lattice is defined using the concept of Weihrauch reducibility, denoted by $f \leq_{\mathrm{W}} g$, and intuitively the meaning is that one can use a realization of g to implement f. A variant of this reducibility has been introduced by Klaus Weihrauch in the 1990s and it has been studied since then (see

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[Ste89, Wei92a, Wei92b, Her96, Bra99, Bra05]). The underlying machinery that allows one to work with different sets such as real numbers $\mathbb{R}$ or other metric spaces X is the theory of representations as it is used in computable analysis [Wei00]. The Weihrauch lattice can be seen as giving a fine structure to the effective Borel hierarchy.

In some sense the Weihrauch lattice is a simple and efficient approach to computable metamathematics. The space that one studies contains the theorems as points (straightforwardly represented by multi-valued functions in the above sense) and the underlying technicalities of data types are hidden and encapsulated in representations. The "user" can fully concentrate on comparing the points (i.e. theorems) in the lattice and one can directly apply methods of computability theory, topology and descriptive set theory without considering any additional models. Despite the fact that no logical system in the proof theoretic sense is used, one obtains a very fine picture of the computational relations of theorems. In particular, the picture is detailed enough to explain the specific computational properties of certain theorems that are left unexplained by some other approaches and yet the picture is in strong correspondence with the results of reverse mathematics, constructive mathematics and proof theory.

In this paper we want to analyze the computational content of the Bolzano-Weierstraß Theorem, which is the statement that any bounded sequence (x_n) of real numbers has a cluster point x. In fact, we will study this theorem more generally for a computable metric space X and then the formulation reads as follows.

Theorem 1.1 (Bolzano-Weierstraß Theorem)**.** *Let X be a metric space. Any sequence (x_n) in X with a relatively compact range has a cluster point x.*

Here a set is called *relatively compact*, if its closure is compact. The straightforward interpretation of this theorem as a partial multi-valued map is denoted by $\mathsf{BWT}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X$ (see Definition 11.1 for the precise definition). We emphasize that the input sequence (x_n) is just given with the guarantee to have a relatively compact range, but no further input information or bound is provided for this set. We also study the *cluster point problem* CL_X, which is an extension of BWT_X in the sense that the guarantee provided for the input sequence (x_n) is only that it has a cluster point, but the range of the sequence is not necessarily relatively compact. Moreover, we also consider the situation that the sequence has a unique cluster point and then the corresponding restrictions of the above functions are denoted by UBWT_X and UCL_X, respectively.

We mention that the finite versions $\mathsf{BWT}_k = \mathsf{CL}_k$ of the Bolzano-Weierstraß Theorem can be interpreted as an infinite version of the pigeonhole principle (here and in the following we identify the number $k \in \mathbb{N}$ with the set $\{0, 1, ..., k-1\}$):

Theorem 1.2 (Infinite Pigeonhole Principle)**.** *In every sequence (x_n) in $k^{\mathbb{N}}$ some element $i < k$ occurs infinitely often.*

Hence, these principles are worth being studied by themselves and our result, mentioned above, shows that the strength of these principles grows in the Weihrauch lattice with k. In [BG11a] we have classified the Baire Category Theorem $\mathsf{BCT} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$ and this theorem can be interpreted as another infinite version of a pigeonhole principle (every "large" metric space cannot be decomposed into countably many "small" portions).

It turns out that the *derivative* or *jump* f' of a multi-valued function is a very useful tool to study higher levels of the Weihrauch lattice. Essentially, it is the counterpart of the Turing jump in the Weihrauch lattice. Intuitively, the derivative f' of f is just the same function, but with weaker input information. The original

information is replaced by a sequence that converges to it. This makes f' usually much harder to compute than f. We introduce and study the derivative and we show that the cluster point problem is the derivative of closed choice C_X, i.e. $\mathsf{C}'_X \equiv_{\mathrm{W}} \mathsf{CL}_X$ and analogously the Bolzano-Weierstraß Theorem is the derivative of compact choice K_X, i.e. $\mathsf{K}'_X \equiv_{\mathrm{W}} \mathsf{BWT}_X$. Hence, the cluster point problem and the Bolzano-Weierstraß Theorem play a role on the third level of the Weihrauch lattice that is analogous to the role of closed and compact choice on the second level. Our further main results on the cluster point problem and the Bolzano-Weierstraß Theorem can be summarized as follows (we discuss the mentioned notions of computability in Section 8):

(1) The Bolzano-Weierstraß Theorem BWT_X is relatively independent of the underlying metric space X. If X is a computable metric space that contains an embedded copy of Cantor space, then $\mathsf{BWT}_X \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$. In particular, we obtain $\mathsf{BWT}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}^n} \equiv_{\mathrm{W}} \mathsf{BWT}_{[0,1]} \equiv_{\mathrm{W}} \mathsf{BWT}_{\ell_2}$.
(2) The finite versions of the Bolzano-Weierstraß Theorem BWT_n yield a proper hierarchy of principles: $\mathsf{BWT}_2 <_{\mathrm{W}} \mathsf{BWT}_3 <_{\mathrm{W}} ... <_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$.
(3) The Bolzano-Weierstraß Theorem on reals is the jump of Weak Kőnig's Lemma, i.e. $\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{WKL}'$.
(4) The Bolzano-Weierstraß Theorem $\mathsf{BWT}_{\mathbb{R}}$ is complete for functions f that are *weakly limit computable*. These are functions that can be represented as composition $f = g \circ h$ of a weakly computable function g and a limit computable h.
(5) The unique cluster point problem $\mathsf{UCL}_{\mathbb{R}}$ is complete for functions f that are *limit computable with finitely many mind changes*. These are functions that can be represented as composition $f = g \circ h$ of a function g that is computable with finitely many mind changes and a limit computable h.
(6) The Bolzano-Weierstraß Theorem $\mathsf{BWT}_{\mathbb{R}}$ and the cluster point problem $\mathsf{CL}_{\mathbb{R}}$ are low limit computable, i.e. if a limit computable function g is composed with any function h below the cluster point problem $\mathsf{CL}_{\mathbb{R}}$, then the resulting function $g \circ h$ is still 3–computable (as the cluster point problem $\mathsf{CL}_{\mathbb{R}}$ itself).
(7) The cluster point problem $\mathsf{CL}_{\mathbb{R}}$ is strictly stronger than the Bolzano-Weierstraß Theorem, i.e. $\mathsf{BWT}_{\mathbb{R}} <_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}}$, the unique version $\mathsf{UCL}_{\mathbb{R}}$ and the cluster point problem $\mathsf{CL}_{\mathbb{N}}$ are incomparable with $\mathsf{BWT}_{\mathbb{R}}$.
(8) The unique Bolzano-Weierstraß Theorem $\mathsf{UBWT}_{\mathbb{R}}$ is complete for limit computable functions and $\mathsf{UBWT}_{\mathbb{N}}$ is complete for functions that are computable with finitely many mind changes (the same holds for the contrapositive version AS of $\mathsf{BWT}_{\mathbb{R}}$, which is sometimes called *Anti-Specker Theorem*). Hence, $\mathsf{UBWT}_{\mathbb{N}}$ and AS are equivalent to the Baire Category Theorem BCT.

Figure 1 in the conclusions visualizes these and other results. We briefly describe the further structure of this paper. In the next two sections we summarize some relevant information on the Weihrauch lattice, its algebraic structure and on the closed choice principle C_X. In the following Sections 4-7 we introduce compositional products and the concept of a derivative. The main result on derivatives is Theorem 5.14, which describes the principal ideal generated by a derivative f' as composition of the principal ideals of f and the limit computable functions. We also briefly discuss algebraic properties of the derivative that help to determine derivatives in practice. In Section 8 we introduce classes of functions that can be described by composition of limit computable functions with other functions and we characterize complete elements of these classes using derivatives. In Sections 9-11 we study the cluster point problem and the Bolzano-Weierstraß Theorem and we show that they are derivatives of closed and compact choice, respectively. We derive numerous other properties from these characterizations. In Sections 12-13 we

provide separation results that help to separate certain versions of the cluster point problem and the Bolzano-Weierstraß Theorem from each other. In Section 14-15 we discuss further variants of the Bolzano-Weierstraß Theorem, such as the contrapositive form of the Bolzano-Weierstraß Theorem. Moreover, we compare the cluster point problem with the accumulation point problem. Finally, in the Conclusion we compare our results with other results that have been obtained in constructive analysis, reverse mathematics and proof theory.

2. The Weihrauch Lattice

In this section we briefly recall some basic results and definitions regarding the Weihrauch lattice. The original definition of Weihrauch reducibility is due to Weihrauch and has been studied for many years (see [Ste89, Wei92a, Wei92b, Her96]). Only recently it has been noticed that a certain variant of this reducibility yields a lattice that is very suitable for the classification of mathematical theorems (see [GM09, BG11b, BG11a, Pau10b, BdBP, Pau10a]). The basic reference for all notions from computable analysis is [Wei00]. The Weihrauch lattice is a lattice of multi-valued functions over represented spaces. We briefly recall the definition of a representation.

Definition 2.1 (Representation)**.** A *representation* δ of a set X is a surjective (potentially partial) function $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. A *represented space* (X, δ) is a set X together with a representation δ of it.

In general we use the symbol "$\subseteq$" in order to indicate that a function is potentially partial. Using represented spaces we can define the concept of a realizer. We denote the composition of two (multi-valued) functions f and g either by $f \circ g$ or by fg.

Definition 2.2 (Realizer)**.** Let $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multi-valued function between represented spaces. A *realizer* of f is a function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ satisfying $\delta_Y F(p) \in f\delta_X(p)$ for all $p \in \mathrm{dom}(f\delta_X)$. We use the notation $F \vdash f$ for expressing that F is a realizer of f.

As realizers are single-valued by definition, the statement that some function F is a realizer always implies its single-valuedness. Realizers allow us to transfer the notions of computability and continuity and other notions available for Baire space to any represented space; a function between represented spaces will be called *computable*, if it has a computable realizer, etc. Given two representations δ_1, δ_2 of X, we say that δ_1 is *reducible* to δ_2, if the identity $\mathrm{id} : (X, \delta_1) \to (X, \delta_2)$ is computable. If the identity is computable in both directions, then we write $\delta_1 \equiv \delta_2$ and we say that the representations are *equivalent*. Now we can define Weihrauch reducibility. By $\langle\ ,\ \rangle : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ we denote the standard pairing function, defined by $\langle p, q\rangle(2n) := p(n)$ and $\langle p, q\rangle(2n+1) := q(n)$ for all $p, q \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

Definition 2.3 (Weihrauch reducibility)**.** Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be multi-valued functions between represented spaces. Define $f \leq_{\mathrm{W}} g$, if there are computable functions $K, H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ satisfying $K\langle \mathrm{id}, GH\rangle \vdash f$ for all $G \vdash g$. In this situation we say that f is *Weihrauch reducible* to g. We write $f \leq_{\mathrm{sW}} g$ and we say that f is *strongly Weihrauch reducible* to g if an analogous condition holds, but with the property $KGH \vdash f$ in place of $K\langle \mathrm{id}, GH\rangle \vdash f$.

Here $K\langle \mathrm{id}, GH\rangle(p) = K\langle p, GH(p)\rangle$ for all $p \in \mathbb{N}^{\mathbb{N}}$. Hence the difference between ordinary and strong Weihrauch reducibility is that the "output modificator" K has direct access to the original input in case of ordinary Weihrauch reducibility, but not in case of strong Weihrauch reducibility. In [GM09] it has been proved that $f \leq_{\mathrm{W}} g$

holds if and only if there are computable multi-valued functions $h :\subseteq X \rightrightarrows Z$ and $k :\subseteq X \times W \rightrightarrows Y$ such that $\emptyset \neq k(x, gh(x)) \subseteq f(x)$ for all $x \in \mathrm{dom}(f)$. Similarly, $\leq_{\mathrm{sW}}$ can be characterized using suitable functions h, k with $\emptyset \neq kgh(x) \subseteq f(x)$.

We note that the relations $\leq_{\mathrm{W}}$, $\leq_{\mathrm{sW}}$ and $\vdash$ implicitly refer to the underlying representations, which we will only mention explicitly if necessary. It is known that these relations only depend on the underlying equivalence classes of representations, but not on the specific representatives (see Lemma 2.11 in [BG11b]). The relations $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$ are reflexive and transitive, thus they induce corresponding partial orders on the sets of their equivalence classes (which we refer to as *Weihrauch degrees* or *strong Weihrauch degrees*, respectively). These partial orders will be denoted by $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$ as well. In this way one obtains distributive bounded lattices (for details see [Pau10b] and [BG11b]). We use $\equiv_{\mathrm{W}}$ and $\equiv_{\mathrm{sW}}$ to denote the respective equivalences regarding $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$, and by $<_{\mathrm{W}}$ and $<_{\mathrm{sW}}$ we denote strict reducibility. It is interesting to mention that some variant of the theory of (continuous) Weihrauch degrees has recently been proved to be undecidable (see [KSZ10]) and some initial fragments have been analyzed with respect to computational complexity (see [HS11]).

The Weihrauch lattice is equipped with a number of useful algebraic operations that we summarize in the next definition. We use $X \times Y$ to denote the ordinary set-theoretic *product*, $X \sqcup Y := (\{0\} \times X) \cup (\{1\} \times Y)$ in order to denote *disjoint sums* or *coproducts*, by $\bigsqcup_{i=0}^{\infty} X_i := \bigcup_{i=0}^{\infty} (\{i\} \times X_i)$ we denote the *infinite coproduct*. By X^i we denote the i–fold product of a set X with itself, where $X^0 = \{()\}$ is some canonical singleton. By $X^* := \bigsqcup_{i=0}^{\infty} X^i$ we denote the set of all *finite sequences over* X and by $X^{\mathbb{N}}$ the set of all *infinite sequences over* X. All these constructions have parallel canonical constructions on representations and the corresponding representations are denoted by $[\delta_X, \delta_Y]$ for the product of (X, δ_X) and (Y, δ_Y), $\delta_X \sqcup \delta_Y$ for the coproduct and δ_X^* for the representation of X^* and $\delta_X^{\mathbb{N}}$ for the representation of $X^{\mathbb{N}}$ (see [BG11b, Pau10b, BdBP] for details). We will always assume that these canonical representations are used, if not mentioned otherwise.

Definition 2.4 (Algebraic operations)**.** Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be multi-valued functions on represented spaces. Then we define the following operations:

(1) $f \times g :\subseteq X \times Z \rightrightarrows Y \times W, (f \times g)(x, z) := f(x) \times g(z)$ (product)

(2) $f \sqcap g : X \times Z \rightrightarrows Y \sqcup W, (f \sqcap g)(x, z) := (\{0\} \times f(x)) \cup (\{1\} \times g(z))$ (sum)

(3) $f \sqcup g :\subseteq X \sqcup Z \rightrightarrows Y \sqcup W$, with $(f \sqcup g)(0, x) := \{0\} \times f(x)$ and $(f \sqcup g)(1, z) := \{1\} \times g(z)$ (coproduct)

(4) $f^* : X^* \rightrightarrows Y^*, f^*(i, x) := \{i\} \times f^i(x)$ (finite parallelization)

(5) $\widehat{f} : X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}, \widehat{f}(x_n) := \bigtimes_{i=0}^{\infty} f(x_i)$ (parallelization)

In this definition and in general we denote by $f^i :\subseteq X^i \rightrightarrows Y^i$ the i–th fold product of the multi-valued map f with itself. For f^0 we assume that $X^0 := \{()\}$ is a canonical singleton for each set X and hence f^0 is just the constant operation on that set. It is known that $f \sqcap g$ is the *infimum* of f and g with respect to strong as well as ordinary Weihrauch reducibility (see [BG11b], where this operation was denoted by $f \oplus g$). Correspondingly, $f \sqcup g$ is known to be the *supremum* of f and g (see [Pau10b]). The two operations $f \mapsto \widehat{f}$ and $f \mapsto f^*$ are known to be *closure operators* in the corresponding lattices, which means $f \leq_{\mathrm{W}} \widehat{f}$ and $\widehat{f} \equiv_{\mathrm{W}} \widehat{\widehat{f}}$, and $f \leq_{\mathrm{W}} g$ implies $\widehat{f} \leq_{\mathrm{W}} \widehat{g}$ and analogously for finite parallelization (see [BG11b, Pau10b]). Sometimes, the finite parallelization is written as $f^* := \bigsqcup_{i=0}^{\infty} f^i$. More generally, we use the notation $\bigsqcup_{i=0}^{\infty} f_i :\subseteq \bigsqcup_{i=0}^{\infty} X_i \rightrightarrows \bigsqcup_{i=0}^{\infty} Y_i$ for a sequence

(f_i) of multi-valued functions $f_i :\subseteq X_i \rightrightarrows Y_i$ on represented spaces and then it denotes the operation given by $(\bigsqcup_{i=0}^{\infty} f_i)(i,u) := \{i\} \times f_i(u)$. We mention that all the algebraic operations mentioned in Definition 2.4 preserve (strong) Weihrauch equivalence.

There is some terminology related to these algebraic operations. We say that f is a *a cylinder* if $f \equiv_{\mathrm{sW}} \mathrm{id} \times f$ where $\mathrm{id} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ always denotes the identity on Baire space, if not mentioned otherwise. Cylinders f have the property that $g \leq_{\mathrm{W}} f$ is equivalent to $g \leq_{\mathrm{sW}} f$ (see [BG11b]). We say that f is *idempotent* if $f \equiv_{\mathrm{W}} f \times f$ and *strongly idempotent*, if $f \equiv_{\mathrm{sW}} f \times f$. We say that a multi-valued function on represented spaces is *pointed*, if it has a computable point in its domain. For pointed f and g we obtain $f \sqcup g \leq_{\mathrm{sW}} f \times g$. If $f \sqcup g$ is (strongly) idempotent, then we also obtain the inverse (strong) reduction. The finite prallelization f^* can also be considered as *idempotent closure* as for pointed f one can easily see that idempotency is equivalent to $f \equiv_{\mathrm{W}} f^*$. We call f *parallelizable* if $f \equiv_{\mathrm{W}} \widehat{f}$ and it is easy to see that $\widehat{f}$ is always idempotent. In [BdBP] a multi-valued function on represented spaces has been called *join-irreducible* if $f \equiv_{\mathrm{W}} \bigsqcup_{n \in \mathbb{N}} f_n$ implies that there is some n such that $f \equiv_{\mathrm{W}} f_n$. Analogously, we can define *strong join-irreducibility* using strong Weihrauch reducibility in both instances. The properties of pointedness, (strong) idempotency and (strong) join-irreducibility are all preserved under (strong) equivalence and hence they can be considered as properties of the respective (strong) degrees.

In [BdBP] a large class of multi-valued functions has been identified that is join-irreducible and we will call them *fractals*.[1] Intuitively, a fractal is a function that is able to compute itself in its entirety from the values of a realizer in any small neighbourhood of its domain. Hence a fractal has a computational self-similarity property. In order to express this property formally, we need the following terminology. If $f :\subseteq X \rightrightarrows Y$ is a function between represented spaces, with representation δ of X, then we define f_A for each set $A \subseteq \mathbb{N}^{\mathbb{N}}$ as follows. We let $(X_A, \delta|_A)$ be the represented space with $X_A := \delta(A)$ and the restriction $\delta|_A$ of δ to A. Then $f_A :\subseteq X_A \rightrightarrows Y$ is the restriction of f to (X_A, δ_A). Using this notation we can define (strong) fractals.

Definition 2.5 (Fractals)**.** Let (X, δ_X) and Y be represented spaces. Then a multi-valued function $f :\subseteq X \rightrightarrows Y$ is called a *strong fractal*, if $f \leq_{\mathrm{sW}} f_A$ for each $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that A is clopen and non-empty in $\mathrm{dom}(f\delta_X)$. We call f a *fractal* if the analogous condition holds for $\leq_{\mathrm{W}}$ instead of $\leq_{\mathrm{sW}}$.

One reason for the importance of fractals is that being a fractal is often an easily verifiable condition that implies join-irreducibility.

Proposition 2.6 (Join-irreducibility of fractals)**.** *Each fractal is join-irreducible, each strong fractal is join-irreducible and strongly join-irreducible.*

The version for ordinary fractals has been proved in Lemma 5.5 of [BdBP]. We mention that the analogous statement for strong fractals and strong join-irreducibility has essentially the same proof. Another concept that turns out to be useful for our purposes is the concept of slimness. We recall that for a multi-valued function $f :\subseteq X \rightrightarrows Y$ we call $\mathrm{range}(f) = \bigcup_{x \in \mathrm{dom}(f)} f(x)$ the *range* of f. This range might contain "superfluous" elements and we call multi-valued functions slim that actually use all elements in their range as singletons.

Definition 2.7 (Slim)**.** Let $f :\subseteq X \rightrightarrows Y$ be a multi-valued function. We call f *slim*, if for all $y \in \mathrm{range}(f)$ there is some $x \in \mathrm{dom}(f)$ such that $f(x) = \{y\}$.

[1]In this context the terminology of a *fractal* has been coined by Arno Pauly (personal communication).

Obviously, all single-valued functions are slim, but many multi-valued functions that we are interested in are also slim. As mentioned already in the introduction, we are occasionally interested in the unique variant of a given multi-valued function, a concept that we define now.

Definition 2.8 (Unique variant)**.** Let $f :\subseteq X \rightrightarrows Y$ be a multi-valued function on represented spaces. Then $\mathsf{U}f :\subseteq X \to Y$ is defined as restriction of f with $\mathrm{dom}(\mathsf{U}f) := \{x \in \mathrm{dom}(f) : f(x) \text{ is a singleton}\}$.

Obviously, $\mathsf{U}f$ is just a restriction of f to the inputs with a unique output. We note that for slim f we obtain $\mathrm{range}(f) = \mathrm{range}(\mathsf{U}f)$.

3. Closed Choice

Particularly interesting degrees in the Weihrauch lattice can be defined as variants of closed choice. This operation has been studied in [GM09, BG11b, BG11a, BdBP] and it is known that many notions of computability can be calibrated using the right version of choice. Basically, closed choice means to find a solution, given a description of what does not constitute a solution. Since for closed choice we only consider closed sets of possible solutions, a negative description means to describe the open complement of the solution set. This can be achieved with the representation ψ_- that we describe now.

A *computable metric space* is a triple (X, d, α) such that (X, d) is a metric space and $\alpha : \mathbb{N} \to X$ is some sequence that is dense in X and such that $d \circ (\alpha \times \alpha)$ is a computable sequence of reals. For each computable metric space we can derive a numbering of open rational balls by

$$B_{\langle n,k\rangle} := B(\alpha(n), \overline{k}) := \{x \in X : d(\alpha(n), x) < \overline{k}\},$$

where $\overline{k}$ denotes the k–th rational number with respect to some standard numbering of rationals. Using this notation we obtain a representation $\psi_- : \mathbb{N}^\mathbb{N} \to \mathcal{A}(X)$ of the set $\mathcal{A}(X) := \{A \subseteq X : A \text{ closed}\}$ by

$$\psi_-(p) := X \setminus \bigcup_{i=0}^{\infty} B_{p(i)}.$$

The full space X is captured here as well, as we also consider empty balls $B(\alpha(n), 0)$. Intuitively, a name p of a closed set $A \subseteq X$ is an enumeration of rational open balls (centered in the dense subset) that exhaust the complement of A. The set $\mathcal{A}(X)$ equipped with the representation ψ_- is denoted by $\mathcal{A}_-(X)$ in order to indicate that we are using *negative information*, which describes the complement of the represented set. The computable points in $\mathcal{A}_-(X)$ are called *co-c.e. closed sets*.

Computable metric spaces themselves are typically represented by the *Cauchy representation* $\delta :\subseteq \mathbb{N}^\mathbb{N} \to X$ that is defined by $\delta(p) = x :\iff \lim_{n\to\infty} \alpha p(n) = x$ for all $p \in \mathbb{N}^\mathbb{N}$ such that $d(\alpha(n), \alpha(k)) < 2^{-n}$ for all $k > n$. If not mentioned otherwise, we will assume that computable metric spaces X are represented with the Cauchy representation and $\mathcal{A}_-(X)$ is represented by ψ_- as defined above. Typically we assume that Baire space $\mathbb{N}^\mathbb{N}$ is represented just by the identity $\mathrm{id} : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and Cantor space $\{0, 1\}^\mathbb{N}$ by its corresponding restriction. In particular, any function $f :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is its only realizer up to extensions.

In some cases ψ_- can also be described in simpler terms. For instance for $X = \mathbb{N}$ we can equivalently define $\psi_-(p) := \mathbb{N} \setminus \{n \in \mathbb{N} : (\exists k)\ p(k) = n + 1\}$. Hence p is a ψ_- name for a set $A \subseteq \mathbb{N}$ if p is an enumeration of all elements in the complement of A (where the number 0 is used as a placeholder that indicates no information and allows to represent $\mathbb{N}$ itself). We now define closed choice for the case of computable metric spaces.

Definition 3.1 (Closed Choice)**.** Let X be a computable metric space. Then the *closed choice* operation of this space is defined by

$$\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$$

with $\text{dom}(\mathsf{C}_X) := \{A \in \mathcal{A}_-(X) : A \neq \emptyset\}$.

Intuitively, C_X takes as input a non-empty closed set in negative description (i.e. given by ψ_-) and it produces an arbitrary point of this set as output. Hence, $A \mapsto A$ means that the multi-valued map C_X maps the input $A \in \mathcal{A}_-(X)$ to the set $A \subseteq X$ as a set of possible outputs. We mention a couple of properties of closed choice for specific spaces. It is easy to see that C_X is always pointed and slim (since singletons $\{x\}$ are closed in metric spaces). We recall that by UC_X we denote unique choice. We recall that we identify $k \in \mathbb{N}$ with the set $\{0, 1, ..., k-1\}$ and hence $\mathsf{C}_1 = \mathsf{C}_{\{0\}}$. Correspondingly, we consider $\mathsf{C}_0 = \mathsf{C}_\emptyset$ as the nowhere defined function (of type $\{\emptyset\} \to \emptyset$), despite the fact that $\emptyset$ is not a computable metric space. Moreover, the following is known.

Fact 3.2 (Closed choice)**.** *We obtain the following:*

1. $\mathsf{C}_\mathbb{N}, \mathsf{C}_{\{0,1\}^\mathbb{N}}, \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ *and* $\mathsf{C}_\mathbb{R}$ *are strongly idempotent and strong fractals, hence also strongly join-irreducible,*
2. $\mathsf{C}_{\{0,1\}^\mathbb{N}}$*,* $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ *and* $\mathsf{C}_\mathbb{R}$ *are cylinders (likewise* $\mathsf{UC}_{\{0,1\}^\mathbb{N}}, \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ *and* $\mathsf{UC}_\mathbb{R}$*),*
3. C_1*,* $\mathsf{C}_\mathbb{N}$*,* $\mathsf{C}_{\{0,1\}^\mathbb{N}}$ *and* $\widehat{\mathsf{C}_\mathbb{N}}$ *are complete with respect to Weihrauch reducibility for the classes of multi-valued function on represented spaces that are computable, computable with finitely many mind changes, weakly computable and limit computable, respectively.*
4. $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ *is complete for all effectively Borel measurable single-valued functions on computable Polish spaces.*

These facts were essentially proved in [BG11b, BG11a, BdBP]. In case of (1) an even stronger property than idempotency is known: the principal ideal given by the respective choice principle is closed under composition, see Corollary 7.6 in [BdBP]. In Corollary 5.6 of [BdBP] the claims on fractals were proved and the statement for strong fractals follows analogously. The claim on cylinders of choice was proved in Proposition 8.11 in [BdBP], except for $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$, for which it follows easily. The related extra claims for unique choice can be proved correspondingly. The statements (3) and (4) have been proved in [BdBP].

The omniscience principles LPO and LLPO turned out to be very useful and they are closely related to the closed choice. We recall the definitions (see [BG11b] for more details).

Definition 3.3 (Omniscience principles)**.** We define:

- $\mathsf{LPO} : \mathbb{N}^\mathbb{N} \to \mathbb{N}, \quad \mathsf{LPO}(p) = \begin{cases} 0 & \text{if } (\exists n \in \mathbb{N})\ p(n) = 0 \\ 1 & \text{otherwise} \end{cases},$
- $\mathsf{LLPO} :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}, \mathsf{LLPO}(p) \ni \begin{cases} 0 & \text{if } (\forall n \in \mathbb{N})\ p(2n) = 0 \\ 1 & \text{if } (\forall n \in \mathbb{N})\ p(2n+1) = 0 \end{cases},$

where $\text{dom}(\mathsf{LLPO}) := \{p \in \mathbb{N}^\mathbb{N} : p(k) \neq 0 \text{ for at most one } k\}$.

It is easy to see that $\mathsf{C}_2 \equiv_{\mathrm{sW}} \mathsf{LLPO}$. Closed choice can be used to characterize the computational content of many theorems. By $\mathsf{WKL} :\subseteq \text{Tr} \rightrightarrows \{0,1\}^\mathbb{N}$ we denote the formalization of Weak Kőnig's Lemma, i.e. Tr denotes the set of binary trees represented via characteristic functions, dom(WKL) is the set of all infinite binary trees and $\mathsf{WKL}(T)$ is the set of all infinite paths in a given infinite tree $T \in \text{Tr}$ (see [BG11b] and [GM09] where WKL was originally introduced under the name Path_2). By HBT we denote the formalization of the Hahn-Banach Theorem (see [GM09] for details).

Fact 3.4 (Weak Kőnig's Lemma)**.** $\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{HBT} \equiv_{\mathrm{sW}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \widehat{\mathsf{LLPO}}$.

The equivalence $\mathsf{WKL} \equiv_{\mathrm{sW}} \widehat{\mathsf{LLPO}}$ was proved in Theorem 8.2 of [BG11b], the equivalence $\mathsf{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \widehat{\mathsf{LLPO}}$ was proved in Theorem 8.5 of [BG11b]. In Proposition 6.5 of [BG11b] it was proved that $\widehat{\mathsf{LLPO}}$ and hence WKL are cylinders. The equivalence $\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{HBT}$ was proved in [GM09] and the proof even shows $\mathsf{WKL} \leq_{\mathrm{sW}} \mathsf{HBT}$. The other direction holds with respect to strong reducibility, since WKL is a cylinder. This also shows that HBT is a cylinder.

Another important equivalence class is the class of choice $\mathsf{C}_{\mathbb{N}}$ on natural numbers, which turned out to be equivalent to the Baire Category Theorem BCT and to the limit operation $\lim_{\mathbb{N}}$ on natural numbers. By $\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, p_2, ...\rangle \mapsto \lim_{i\to\infty} p_i$ we denote the usual limit operation on Baire space (with the input sequence encoded in a single sequence) and by $\lim_{\Delta}$ we denote the restriction of lim to the limit with respect to the discrete topology on $\mathbb{N}^{\mathbb{N}}$. It is easy to see that lim and $\lim_{\Delta}$ are cylinders (see below). In general, we denote by $\lim_X :\subseteq X^{\mathbb{N}} \to X$ the ordinary limit operation of a metric space X. We mention some known facts.

Fact 3.5 (Limit)**.** $\lim \equiv_{\mathrm{sW}} \lim_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \lim_{\mathbb{R}} \equiv_{\mathrm{sW}} \widehat{\mathsf{LPO}} \equiv_{\mathrm{sW}} \widehat{\lim_{\mathbb{N}}}$ *and all the mentioned functions are cylinders.*

The claim can be derived from Proposition 9.1 in [Bra05], Corollary 6.4 and Proposition 6.5 in [BG11b] and the equivalence $\lim \equiv_{\mathrm{sW}} \widehat{\lim_{\mathbb{N}}}$ can easily be seen directly.

Fact 3.6 (Baire Category)**.** $\mathsf{BCT} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \lim_{\mathbb{N}} \equiv_{\mathrm{W}} \lim_{\Delta} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{R}}$.

The equivalence $\mathsf{BCT} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$ has been proved in Theorem 5.2 of [BG11a]. The equivalence $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{R}}$ has been proved in Corollary 6.4 of [BdBP], the equivalence $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \lim_{\Delta}$ has been proved in Corollary 7.11 of [BdBP]. In Proposition 6.2 of [BdBP] it was proved that $\mathsf{UC}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$. The equivalence of $\lim_{\mathbb{N}}$ and $\mathsf{C}_{\mathbb{N}}$ is discussed in Proposition 3.8 below.

Although the above equivalence describes a single Weihrauch degree, this degree decomposes into a number of interesting strong degrees. Firstly, we mention that $\lim_{\Delta}$ and $\mathsf{UC}_{\mathbb{R}}$ are cylinders. This is easy to see in case of $\lim_{\Delta}$ (using the normal pairing function on Baire space, we obtain $\langle q, \lim_{\Delta}(p_i)\rangle = \lim_{\Delta}\langle q, p_i\rangle$.) In case of $\mathsf{UC}_{\mathbb{R}}$, this can be proved as for $\mathsf{C}_{\mathbb{R}}$, see Fact 3.2.

Fact 3.7. $\lim_{\Delta} \equiv_{\mathrm{sW}} \mathsf{UC}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}} \times \mathrm{id}$ *and* $\lim_{\Delta}$ *and* $\mathsf{UC}_{\mathbb{R}}$ *are cylinders.*

Since the other four functions mentioned in Fact 3.6 cannot be cylinders (for mere cardinality reasons of the output), it follows that they are not in the same strong degree. We strengthen here the above result by proving that at least three of the above functions are in the same strong degree.

Proposition 3.8. $\mathsf{UC}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \lim_{\mathbb{N}}$.

Proof. It is clear that $\mathsf{UC}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$. It can easily be seen that also $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{sW}} \lim_{\mathbb{N}}$. To this end a sequence $p \in \mathbb{N}^{\mathbb{N}}$ such that $\{n : n+1 \in \mathrm{range}(p)\} = \mathbb{N} \setminus A$, for a non-empty $A \subseteq \mathbb{N}$, is scanned for the least number larger than 0 that is missing. This number is written to the output repeatedly, until it appears in the input. Then the number is replaced by the next missing number. Eventually this process will converge, since A is non-empty. It is clear that $\lim_{\mathbb{N}}$, applied to the output, yields a number i such that $i-1 \in A$.

We prove the reduction $\lim_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{UC}_{\mathbb{N}}$. Given a sequence (n_i) that converges to n, we generate a sequence $p \in \mathbb{N}^{\mathbb{N}}$ such that $A := \mathbb{N} \setminus \{n : n+1 \in \mathrm{range}(p)\}$ has a single element. For this purpose we scan the input sequence (n_i) and seeing

the first element n_0 we start to generate a list of all numbers $\langle m,k\rangle+1$ except $\langle n_0,0\rangle+1$. At stage $i+1$, if the next element n_{i+1} on the input is identical to the previous n_i, then we just continue with this process. If some new element $n_{i+1}\neq n_i$ appears on the input side, then we add the number $\langle n_i,k\rangle+1$ that was previously left out to the output, and we continue enumerating all numbers $\langle m,k\rangle+1$ except for $\langle n_{i+1},k\rangle+1$, where k is least such that the corresponding number was not yet enumerated. Since the input sequence converges to n, it is eventually constant with value n and the process will end enumerating a name of the set $A=\{\langle n,k\rangle\}$ for some k. Unique choice $\mathsf{UC}_{\mathbb{N}}$ applied to this set yields $\langle n,k\rangle$ and the projection to the first component is the limit n of the input sequence. □

The finer characterization provided by Proposition 3.8 is useful for the classification of the Bolzano-Weierstraß Theorem $\mathsf{UBWT}_{\mathbb{N}}$, which, in fact, is identical to $\lim_{\mathbb{N}}$. Finally, we mention that in Corollary 8.12 and Theorem 8.10, both in [BdBP], a uniform version of the Low Basis Theorem was proved. We state this result for further reference here as well.

Fact 3.9 (Uniform Low Basis Theorem)**.** $\mathsf{C}_{\mathbb{R}}\leq_{\mathrm{sW}}\mathfrak{L}$.

We recall that $\mathfrak{L}:=J^{-1}\circ\lim$. Here $J:\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}},p\mapsto p'$ denotes the *Turing jump operator*, where p' is the Turing jump[2] of $p\in\mathbb{N}^{\mathbb{N}}$. We point out that we consider J as a set-theoretic function and not as an operator on Turing degrees. In the former sense it is easily seen to be injective (in the latter sense it is known not to be injective). A point $p\in\mathbb{N}^{\mathbb{N}}$ is *low* if and only if there is a computable q such that $\mathfrak{L}(q)=p$. The classical Low Basis Theorem of Jockusch and Soare [JS72] states that any non-empty co-c.e. closed set $A\subseteq\{0,1\}^{\mathbb{N}}$ has a low member and Fact 3.9 can be seen as a uniform version of this result (see [BdBP] for a further discussion of this theorem).

4. Compositional Products

We define two types of compositional products, one with respect to ordinary Weihrauch reducibility and the other one with respect to strong Weihrauch reducibility.

Definition 4.1 (Compositional product)**.** Let f and g be multi-valued functions on represented spaces. Then we define the *compositional product*

$$f*g:=\sup\{f_0\circ g_0:f_0\leq_{\mathrm{W}}f\text{ and }g_0\leq_{\mathrm{W}}g\}.$$

Only compositions $f_0\circ g_0$ with compatible types are considered here. The supremum is understood with respect to $\leq_{\mathrm{W}}$. By $f*_{\mathrm{s}}g$ we denote the *strong compositional product* where both reductions are replaced by $\leq_{\mathrm{sW}}$ and the supremum is also understood with respect to $\leq_{\mathrm{sW}}$.

We point out that the compositional product $f*g$, if it exists, is a Weihrauch degree, not just a specific multi-valued function. Nevertheless, we treat it in the following as if it is some representative of its equivalence class. This will not lead to any confusion, mainly because the compositional product is monotone, as the next result shows.

Lemma 4.2 (Monotonicity)**.** *Let f_1,f_2,g_1 and g_2 be multi-valued functions on represented spaces. If f_1*g_1 and f_2*g_2 exist, then the following holds:*

$$f_1\leq_{\mathrm{W}}f_2\text{ and }g_1\leq_{\mathrm{W}}g_2\Longrightarrow f_1*g_1\leq_{\mathrm{W}}f_2*g_2.$$

[2] More formally, the Turing jump $p'\in\{0,1\}^{\mathbb{N}}$ of a sequence $p\in\mathbb{N}^{\mathbb{N}}$ can be considered as the characteristic function of the ordinary Turing jump of the set $\mathrm{graph}(p)\subseteq\mathbb{N}^2$, but we will make no technical use of this definition.

An analogous result holds for strong Weihrauch reducibility $\leq_{\mathrm{sW}}$ and the strong compositional product $*_{\mathrm{s}}$.

Proof. If $f_1 \leq_{\mathrm{W}} f_2$ and $g_1 \leq_{\mathrm{W}} g_2$, then we obtain by transitivity

$$\{f_0 \circ g_0 : f_0 \leq_{\mathrm{W}} f_1 \text{ and } g_0 \leq_{\mathrm{W}} g_1\} \subseteq \{f_0 \circ g_0 : f_0 \leq_{\mathrm{W}} f_2 \text{ and } g_0 \leq_{\mathrm{W}} g_2\},$$

which implies the claim. □

The next result shows that the compositional product is related to the ordinary product of two multi-valued function.

Lemma 4.3 (Products and compositional products)**.** *Let f and g be multi-valued maps on represented spaces. If $f * g$ exists, then $f \times g \leq_{\mathrm{W}} f * g$. If $f *_{\mathrm{s}} g$ exists and f, g are cylinders, then $f \times g \leq_{\mathrm{sW}} f *_{\mathrm{s}} g$.*

Proof. Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$. Then $f \times \mathrm{id}_Z \leq_{\mathrm{W}} f$ and $\mathrm{id}_X \times g \leq_{\mathrm{W}} g$. Then we obtain $f \times g = (f \times \mathrm{id}_Z) \circ (\mathrm{id}_X \times g) \leq_{\mathrm{W}} f * g$. The second claim is proved analogously. □

One could ask whether the reduction $f \times g \leq_{\mathrm{W}} f * g$ can be strengthened to an equivalence or to a strict reduction in general. We provide two examples that show that the equivalence might or might not hold and we provide another example that shows that the compositional product cannot be exchanged with parallelization.

Example 4.4. We obtain the following:

(1) $\lim \times \lim \equiv_{\mathrm{W}} \lim <_{\mathrm{W}} \lim \circ \lim \equiv_{\mathrm{W}} \lim * \lim$,
(2) $\mathsf{C}_{\{0,1\}^{\mathbb{N}}} \times \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}} * \mathsf{C}_{\mathbb{N}}$,
(3) $\widehat{\mathsf{C}_{\mathbb{N}} * \mathsf{C}_{\mathbb{N}}} \equiv_{\mathrm{W}} \widehat{\mathsf{C}_{\mathbb{N}}} <_{\mathrm{W}} \widehat{\mathsf{C}_{\mathbb{N}}} * \widehat{\mathsf{C}_{\mathbb{N}}}$.

The correctness of these examples follows from results in [BG11b] and Corollaries 4.9 and 7.6 in [BdBP].

5. Derivatives

Now we define the *jump* or *derivative* of a Weihrauch degree. To some extent this concept yields an analogue of the Turing jump for Weihrauch reducibility. We use the jump $\delta' := \delta \circ \lim$ of a representation for this purpose, as it has been used by Ziegler [Zie07].

Definition 5.1 (Derivative)**.** Let $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multi-valued function on represented spaces. Then the *derivative* or *jump* f' of f is the function $f :\subseteq (X, \delta'_X) \rightrightarrows (Y, \delta_Y)$, i.e. the same function, but defined on the input space with the jump of the original representation. By $f^{(n)}$ we denote the n–th derivative of f for $n \in \mathbb{N}$, which is defined inductively by $f^{(0)} := f$ and $f^{(n+1)} := (f^{(n)})'$.

The intuition behind this definition is that the derivative of a function f is the same function, but with a different input representation.[3] The derivative δ' as input representation yields less information about the input than the original representation δ, namely only a sequence that converges to some input with respect to δ. Having less input information makes f' potentially harder to realize than f. For functions $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ the derivative can be determined easily.

Lemma 5.2. *Let $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a function. Then $F' \equiv_{\mathrm{sW}} F \circ \lim$.*

[3]We note that the jump operation defined here does not commute with Turing jumps under the embedding of Turing degrees defined in [BG11b]. For the latter purpose one would have to define a different jump operation on Weihrauch degrees that is applicable on the output side.

This follows from the fact that F is its unique realizer (with respect to the identity as representation of Baire space) and hence $F \circ \lim$ is the unique realizer of F'. We mention a couple of examples. By $\mathrm{id}_X : X \to X$ we denote the identity of X (we recall our convention $\mathrm{id} = \mathrm{id}_{\mathbb{N}^\mathbb{N}}$).

Example 5.3. We obtain the following:

(1) $\mathsf{C}_0' \equiv_{\mathrm{sW}} \mathsf{C}_0$,
(2) $\mathsf{C}_1' \equiv_{\mathrm{sW}} \mathsf{C}_1$,
(3) $\mathrm{id}_2' \equiv_{\mathrm{sW}} \lim_2$,
(4) $\mathrm{id}_\mathbb{N}' \equiv_{\mathrm{sW}} \lim_\mathbb{N}$,
(5) $\mathrm{id}' \equiv_{\mathrm{sW}} \lim$,
(6) $\lim' \equiv_{\mathrm{sW}} \lim \circ \lim$,
(7) $(J^{-1})' \equiv_{\mathrm{sW}} J^{-1} \circ \lim = \mathfrak{L}$,
(8) $\mathfrak{L}' \equiv_{\mathrm{sW}} J^{-1} \circ \lim'$.

We point out that this example in particular shows that

$$\mathsf{C}_1' \equiv_{\mathrm{sW}} \mathsf{C}_1 <_{\mathrm{W}} \mathrm{id}_\mathbb{N}' \equiv_{\mathrm{sW}} \lim_\mathbb{N} <_{\mathrm{W}} \lim \equiv_{\mathrm{sW}} \mathrm{id}'$$

despite the fact that $\mathsf{C}_1 \equiv_{\mathrm{W}} \mathrm{id}_\mathbb{N} \equiv_{\mathrm{W}} \mathrm{id}$. Hence, we cannot expect that derivatives are monotone with respect to $\leq_{\mathrm{W}}$ at all. We will see this again in Example 9.12. In some sense the derivative can "amplify" small differences between functions (even from the same Weihrauch degree) to substantial differences between their derivatives. In Example 6.4 we will see that also the opposite can happen: functions from different Weihrauch degrees can have derivatives of even the same strong Weihrauch degree. However, Proposition 5.6 shows that the amplification of differences cannot happen for functions from the same strong Weihrauch degree: derivatives are monotone with respect to strong Weihrauch reducibility $\leq_{\mathrm{sW}}$. In order to prove this we first provide a technical lemma that relates realizers of functions to realizers of their derivatives. We mention that we will use this result several times and our proof uses the Axiom of Choice.

Lemma 5.4 (Jump realization)**.** *Let f and g be multi-valued functions on represented spaces. Let $H, K :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be functions. Then the following are equivalent:*

(1) $HG \lim K \vdash f$ for all $G \vdash g$,
(2) $HFK \vdash f$ for all $F \vdash g'$.

Proof. We consider $g :\subseteq Z \rightrightarrows W$ and the representation δ_Z of Z. Let us assume that $HG \lim K \vdash f$ for all $G \vdash g$ and let $F \vdash g'$. Let $p \in \mathbb{N}^\mathbb{N}$ be a name for some point in $\mathrm{dom}(f)$. Then $\lim K(p) \in \mathrm{dom}(g\delta_Z)$ and hence $K(p) \in \mathrm{dom}(g\delta_Z')$. By the Axiom of Choice there exists some $G \vdash g$. This G can be modified on input $\lim K(p)$ in order to obtain a $G_p \vdash g$ with $G_p \lim K(p) = FK(p)$. This implies $HFK(p) = HG_p \lim K(p)$ and hence the claim follows.

For the other direction we note that for $G \vdash g$ we have $G \lim \vdash g'$, which implies the claim. $\square$

Now we mention a normal form result for limit computable functions, the proof of which is easy and has been provided in [Bra07]. We call a function $H :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *transparent* if for every computable $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ there exists a computable $G :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $FH = HG$ holds.[4]

Fact 5.5. *Let $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be a function. Then the following are equivalent:*

(1) F is limit computable (i.e. $F \leq_{\mathrm{W}} \lim$),

[4]Matthew de Brecht has introduced the name "jump operator" for transparent functions, which we do not use here in order to avoid confusion with the jump.

(2) $F = \lim G$ *for some computable* $G :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$,
(3) $F = GJ$ *for some computable* $G :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.

In particular, $\lim$ *and* J^{-1} *are transparent.*

It is clear that the class of transparent functions contains the identity and is closed under composition.

Now we can formulate and prove our main result on monotonicity, which relies on the Axiom of Choice (via the Jump Realization Lemma 5.4). From now on we will not mention such indirect references to the Axiom of Choice any longer.

Proposition 5.6 (Monotonicity of derivatives)**.** *Let f and g be multi-valued functions on represented spaces. We obtain:*

(1) $f \leq_{\mathrm{sW}} f'$,
(2) $f \leq_{\mathrm{sW}} g \Longrightarrow f' \leq_{\mathrm{sW}} g'$.

Proof. (1) The computable function $K : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined by $K(p) = \langle p, p, p, ...\rangle$ satisfies $\lim \circ K = \mathrm{id}$. Taking $H = \mathrm{id}$ we have $HF \lim K = F$ for every $F \vdash f$. By Lemma 5.4 it follows that $HGK \vdash f$ for every $G \vdash f'$ and hence $f \leq_{\mathrm{sW}} f'$.
(2) Let us now assume $f \leq_{\mathrm{sW}} g$. Then there are computable functions $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

(a) $HGK \vdash f$ for all $G \vdash g$.

It follows that

(b) $HGK \lim \vdash f'$ for all $G \vdash g$.

By Fact 5.5 lim is transparent and it follows that there is some computable K_0 such that $\lim K_0 = K \lim$. This implies that

(c) $HG \lim K_0 \vdash f'$ for all $G \vdash g$.

By the Jump Realization Lemma 5.4 we obtain

(d) $HEK_0 \vdash f'$ for all $E \vdash g'$.

This means $f' \leq_{\mathrm{sW}} g'$. □

This result allows us to extend the concept of a derivative from single functions to entire strong Weihrauch degrees. The derivative of a strong Weihrauch degree is just the strong equivalence class of the derivative of some representative of the original degree. The previous proposition guarantees that the result does not depend on the representative. Altogether, the behaviour of the derivative with respect to strong Weihrauch reducibility is similar to the behaviour of the Turing jump with respect to Turing reducibility.

Next we want to understand how derivatives interact with the algebraic structure of the lattice.

Proposition 5.7 (Algebraic properties of the derivative)**.** *Let f and g be multi-valued functions on represented spaces. Then we obtain*

(1) $f \circ g' = (f \circ g)'$,
(2) $f' \times g' \equiv_{\mathrm{sW}} (f \times g)'$,
(3) $\widehat{f'} \equiv_{\mathrm{sW}} (\widehat{f})'$,
(4) $f' \sqcap g' \equiv_{\mathrm{sW}} (f \sqcap g)'$,
(5) $f' \sqcup g' \leq_{\mathrm{sW}} (f \sqcup g)'$,
(6) $f'^{*} \leq_{\mathrm{sW}} f^{*\prime}$,
(7) $\mathsf{U}(f') = (\mathsf{U}f)'$.

Proof. The first claim (1) follows directly from the definition. Let (X, δ_X) and (Y, δ_Y) now be represented spaces. Then $[\delta_X, \delta_Y]' \equiv [\delta'_X, \delta'_Y]$ and $(\delta_X^{\mathbb{N}})' \equiv (\delta'_X)^{\mathbb{N}}$ is easy to see and has been proved in [Bra07]. Hence claims (2)–(4) follow. Due to monotonicity of the derivative and the fact that $\sqcup$ is the supremum with respect to

$\leq_{\rm sW}$, we obtain $f' \leq_{\rm sW} (f \sqcup g)'$ and $g' \leq_{\rm sW} (f \sqcup g)'$ and hence $f' \sqcup g' \leq_{\rm sW} (f \sqcup g)'$. We obtain $f'^* = \bigsqcup_{i=0}^{\infty} (f')^i \equiv_{\rm sW} \bigsqcup_{i=0}^{\infty} (f^i)' =: h$ with the help of (2). If $G \vdash f^{*\prime} = (\bigsqcup_{i=0}^{\infty} f^i)'$, then H with $H\langle n,p\rangle := G\langle s(n),p\rangle$ is a realizer of h, where $s : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ is the computable function that maps any number n to the constant sequence with value n. Hence, we obtain $f'^* \equiv_{\rm sW} h \leq_{\rm sW} f^{*\prime}$. The identity $(\mathsf{U}f)' = \mathsf{U}(f')$ follows directly from the definition since $\mathsf{U}f$ is a restriction of f. □

Another useful algebraic property of derivatives is that they are necessarily join-irreducible. This follows with Proposition 2.6 from the fact that they are strong fractals.

Proposition 5.8 (Join-irreducibility of derivatives)**.** *Let f be a multi-valued function on represented spaces. Then f' is a strong fractal and hence strongly join-irreducible and join-irreducible.*

Proof. We assume that $f :\subseteq X \rightrightarrows Y$, where δ_X is the representation of X. Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be clopen and non-empty in $\mathrm{dom}(f\delta_X')$. Then there is some word $w \in \mathbb{N}^*$ with $\emptyset \neq w\mathbb{N}^{\mathbb{N}} \cap \mathrm{dom}(f\delta_X') \subseteq A$. A name $p = \langle p_0, p_1, p_2, ...\rangle$ with respect to δ_X' consists of a sequence (p_n) that converges to a name with respect to δ_X. We can find $q_0, ..., q_n \in \mathbb{N}^{\mathbb{N}}$ such that $w \sqsubseteq q_p := \langle q_0, ..., q_n, p_0, p_1, p_2, ...\rangle$ for all $p = \langle p_0, p_1, ...\rangle$. Since $\lim(q_p) = \lim(p)$ and the function $K : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, p \mapsto q_p$ is computable, we immediately obtain $F\delta_X'(p) = F\delta_X'K(p)$ for all $F \vdash f$. This proves that f is a strong fractal and hence it is join-irreducible and strongly join-irreducible by Proposition 2.6. □

In particular, this result allows to show that certain degrees are not derivatives. For instance $\mathsf{C}_{\{0,1\}^{\mathbb{N}}} \sqcup \mathsf{C}_{\mathbb{N}}$ is a join of two incomparable multi-valued functions (see Section 4 in [BG11b]) and hence it is neither join irreducible nor strongly join-irreducible and hence not a derivative.

Example 5.9. There is no multi-valued function f on represented spaces such that $f' \equiv_{\rm W} \mathsf{C}_{\{0,1\}^{\mathbb{N}}} \sqcup \mathsf{C}_{\mathbb{N}}$.

The similar Example 12.9 shows that the result on coproducts in Proposition 5.7 cannot be strengthened to equivalence in general.

A consequence of Proposition 5.7 is that the derivative f' of a cylinder f is a cylinder again. We can even say more than this.

Corollary 5.10. *Let f be a multi-valued function on represented spaces. Then*

$$(f \times \mathrm{id})' \equiv_{\rm sW} f' \times \lim .$$

In particular, if f is a cylinder, then $f' \equiv_{\rm sW} f' \times \lim$ and f' is a cylinder.

Now we can also conclude that for cylinders the derivative is monotone with respect to ordinary Weihrauch reducibility. This is because for cylinders g also g' is a cylinder and strong reducibility to a cylinder is equivalent to ordinary reducibility.

Corollary 5.11. *Let f and g be multi-valued functions on represented spaces and let g be a cylinder. We obtain that $f \leq_{\rm W} g$ implies $f' \leq_{\rm W} g'$.*

This implies that a meaningful definition of a derivative of a Weihrauch degree with representative f is the Weihrauch degree of the derivative of $f \times \mathrm{id}$. Since $f \times \mathrm{id} \equiv_{\rm W} f$ and $f \times \mathrm{id}$ is a cylinder, this definition does not depend on the representative f.

From Proposition 5.7 we can also derive the following result on idempotency.

Corollary 5.12. *Let f be a multi-valued function on represented spaces.*

(1) If f is strongly idempotent, then f' is strongly idempotent too.

(2) If f is idempotent and a cylinder, then f' is idempotent too.

This follows since $f \times f \leq_{\mathrm{sW}} f$ implies $f' \times f' \equiv_{\mathrm{sW}} (f \times f)' \leq_{\mathrm{sW}} f'$.

The following theorem characterizes derivatives in terms of compositions with limit computable functions. For two multi-valued functions $f_1, f_2 :\subseteq X \rightrightarrows Y$ we write $f_1 \sqsupseteq f_2$ if $\mathrm{dom}(f_1) \subseteq \mathrm{dom}(f_2)$ and $f_1(x) \supseteq f_2(x)$ for all $x \in \mathrm{dom}(f_1)$. It is worth mentioning that $f_1 \sqsupseteq f_2$ implies $f_1 \leq_{\mathrm{sW}} f_2$ and that for a multi-valued function $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ the property $F \vdash f$ is equivalent to $f \sqsupseteq \delta_Y F \delta_X^{-1}$. We will use the following observation in the proof of Theorem 5.14.

Lemma 5.13. *Let $f, g :\subseteq X \rightrightarrows Y$ be multi-valued functions on represented spaces. Then the following are equivalent:*

(1) $f \sqsupseteq g$,
(2) $F \vdash f$ for all $F \vdash g$.

The proof follows immediately, using the Axiom of Choice.

Theorem 5.14 (Derivatives)**.** *Let f and g be multi-valued functions on represented spaces. If g is a cylinder, then the following are equivalent:*

(1) $f \leq_{\mathrm{W}} g'$,
(2) $f = g_0 \circ l_0$ for some $g_0 \leq_{\mathrm{W}} g$ and $l_0 \leq_{\mathrm{W}} \lim$.

If g is not necessarily a cylinder, then an analogous equivalence holds with $\leq_{\mathrm{sW}}$ in place of $\leq_{\mathrm{W}}$ and with either $\sqsupseteq$ or $\leq_{\mathrm{sW}}$ instead of $=$.

Proof. "(2)$\Longrightarrow$(1)" Let $f \leq_{\mathrm{sW}} g_0 \circ l_0$ with $g_0 \leq_{\mathrm{sW}} g$ and $l_0 \leq_{\mathrm{sW}} \lim$. (If g is a cylinder, then this follows from the assumption as stated in (2) above. Otherwise, it follows from $f \sqsupseteq g_0 \circ l_0$.) Then there are computable $H, K, H_1, K_1, H_2, K_2 :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

(a) $HFK \vdash f$ for all $F \vdash g_0 \circ l_0$,
(b) $H_1 R K_1 \vdash g_0$ for all $R \vdash g$,
(c) $H_2 \lim K_2 \vdash l_0$ (where lim is the only realizer of itself up to extension).

In particular, by combination of these properties

(d) $H_1 R K_1 H_2 \lim K_2 \vdash g_0 \circ l_0$ for all $R \vdash g$,
(e) $H H_1 R K_1 H_2 \lim K_2 K \vdash f$ for all $R \vdash g$.

Since $K_1 H_2 \lim K_2 K$ is limit computable, there is a computable K_3 such that $\lim K_3 = K_1 H_2 \lim K_2 K$ by Fact 5.5. Moreover, $H_3 = H H_1$ is computable. We obtain by simplification of (e) that

(f) $H_3 R \lim K_3 \vdash f$ for all $R \vdash g$.

By the Jump Realization Lemma 5.4 this implies

(g) $H_3 S K_3 \vdash f$ for all $S \vdash g'$.

This implies $f \leq_{\mathrm{sW}} g'$.

"(1)$\Longrightarrow$(2)" We consider $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ with represented spaces (X, δ_X), (Y, δ_Y), (Z, δ_Z) and (W, δ_W). Let us now assume that $f \leq_{\mathrm{sW}} g'$. That means that there are computable $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

(h) $HSK \vdash f$ for all $S \vdash g'$.

Now we consider the functions $g_1 := \delta_Y H :\subseteq \mathbb{N}^{\mathbb{N}} \to Y$, $g_2 := \delta_W^{-1} g \delta_Z :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$, $g_0 := g_1 g_2 :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows Y$ and $l_0 := \lim K \delta_X^{-1} :\subseteq X \rightrightarrows \mathbb{N}^{\mathbb{N}}$. We claim that $f \sqsupseteq g_0 \circ l_0$ and $g_0 \leq_{\mathrm{sW}} g$ and $l_0 \leq_{\mathrm{sW}} \lim$. Firstly, it is clear that $l_0 \leq_{\mathrm{sW}} \lim K \leq_{\mathrm{sW}} \lim$. Secondly, g_2 and g share the same realizers, i.e.

(i) $R \vdash g_2 \iff R \vdash g$,

which implies $g_0 \leq_{\mathrm{sW}} g_2 \equiv_{\mathrm{sW}} g$. Moreover, it also implies that $g_2 \lim$ and g' share the same realizers as well:

(j) $S \vdash g_2 \lim \iff S \vdash g'$.

Together with (h) this implies $\delta_Y H g_2 \lim K(p) \subseteq f\delta_X(p)$ for all $p \in \mathrm{dom}(f\delta_X)$, which means

(k) $F \vdash f$ for all $F \vdash H g_2 \lim K$.

Altogether, we obtain

(l) $F \vdash f \Longleftarrow F \vdash \delta_Y H g_2 \lim K \delta_X^{-1} \Longleftarrow F \vdash g_0 \circ l_0$.

But since $f, g_0 \circ l_0$ are both multi-valued functions from X to Y, this implies $f \sqsupseteq g_0 \circ l_0$ by Lemma 5.13. This proves the claim for the case that g is not necessarily a cylinder.

We now refine the proof for the case that g is a cylinder. Let us hence assume that g is a cylinder and $f \leq_{\mathrm{W}} g'$. That means that there are computable $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

(h') $H\langle \mathrm{id}, SK\rangle \vdash f$ for all $S \vdash g'$.

Now we consider the functions $g_1 := \delta_Y H :\subseteq \mathbb{N}^{\mathbb{N}} \to Y$, $g_2 := \delta_W^{-1} g \delta_Z :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$, $g_3 := g_1 \circ \langle \mathrm{id} \times g_2\rangle \circ \pi^{-1} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows Y$ and $l_0 := \langle \mathrm{id}, \lim K\rangle \delta_X^{-1} :\subseteq X \rightrightarrows \mathbb{N}^{\mathbb{N}}$. Here $\pi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, (p, q) \mapsto \langle p, q\rangle$ denotes the standard pairing function. We claim that $f \sqsupseteq g_3 \circ l_0$ and $g_3 \leq_{\mathrm{W}} g$ and $l_0 \leq_{\mathrm{W}} \lim$. Firstly, it is clear that $l_0 \leq_{\mathrm{W}} \langle \mathrm{id}, \lim K\rangle \leq_{\mathrm{W}} \lim$. Secondly, $\langle \mathrm{id} \times g_2\rangle \circ \pi^{-1}$ and $\mathrm{id} \times g$ share the same realizers, i.e.

(i') $R \vdash \langle \mathrm{id} \times g_2\rangle \circ \pi^{-1} \iff R \vdash \mathrm{id} \times g$,

which implies $g_3 \leq_{\mathrm{W}} \langle \mathrm{id} \times g_2\rangle \circ \pi^{-1} \equiv_{\mathrm{W}} \mathrm{id} \times g \equiv_{\mathrm{W}} g$. Moreover, also g_2 and g share the same realizers, which implies that $g_2 \lim$ and g' share the same realizers as well:

(j') $S \vdash g_2 \lim \iff S \vdash g'$.

Together with (h') this implies $\delta_Y H\langle p, g_2 \lim K(p)\rangle \subseteq f\delta_X(p)$ for all $p \in \mathrm{dom}(f\delta_X)$, which means

(k') $F \vdash f$ for all $F \vdash H\langle \mathrm{id}, g_2 \lim K\rangle$.

Altogether, we obtain

(l') $F \vdash f \Longleftarrow F \vdash \delta_Y H\langle \mathrm{id}, g_2 \lim K\rangle \delta_X^{-1} \Longleftarrow F \vdash g_3 \circ l_0$.

But since $f, g_3 \circ l_0$ are both multi-valued functions from X to Y, this implies $f \sqsupseteq g_3 \circ l_0$ by Lemma 5.13. In this situation we can now replace g_3 by $g_0 \sqsupseteq g_3$ such that $f = g_0 \circ l_0$. This is possible, because g_3 in the composition $g_3 \circ l_0$ has direct access to a name of the original input of l_0, due to the definition of g_3 and l_0. Hence one can just extend $g_3 :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows Y$ in the image as necessary in order to obtain $g_0 :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows Y$ with $f = g_0 \circ l_0$. For any $g_0 \sqsupseteq g_3$ we obtain $g_0 \leq_{\mathrm{W}} g_3 \leq_{\mathrm{W}} g$. □

We mention that the property that g is a cylinder has only been used for the direction (2)$\Longrightarrow$(1). The requirement that g is a cylinder is not superfluous as the example $g = \mathsf{C}_1$ shows. In this case we have $\mathrm{id} \leq_{\mathrm{W}} g$ but $\lim \not\leq_{\mathrm{W}} \mathsf{C}_1 \equiv_{\mathrm{sW}} g'$.

Statement (j) in the proof of Theorem 5.14 provides a kind of a normal form for derivatives. We formulate this more precisely.

Corollary 5.15. *Let g be a multi-valued function on represented spaces. Then $g' \equiv_{\mathrm{sW}} g_0 \circ \lim$ for some $g_0 \equiv_{\mathrm{sW}} g$.*

Another way of reading Theorem 5.14 is that for cylinders g the principal ideal $\{f : f \leq_{\mathrm{W}} g'\}$ of g' coincides with

$$M = \{g_0 \circ l_0 : g_0 \leq_{\mathrm{W}} g \text{ and } l_0 \leq_{\mathrm{W}} \lim\}.$$

In the case of strong reducibility $\leq_{\mathrm{sW}}$ instead of $\leq_{\mathrm{W}}$ and arbitrary g we can only say that the corresponding set M_s is included in the strong principal ideal of g' and any f in the strong principal ideal of g' is represented in M_s by an extension in the

image. In both cases this means that g' is a representative of the supremum of the corresponding set M or M_s, respectively and in case that g is a cylinder it is even the maximum of M.

Corollary 5.16 (Derivatives)**.** *Let g be a multi-valued function on represented spaces. Then $g *_{\rm s} \lim$ exists and $g' \equiv_{\rm sW} g *_{\rm s} \lim$. If g is a cylinder, then $g * \lim$ exists and $g' \equiv_{\rm W} g * \lim$.*

We point out that the formulation in this corollary is a slight abuse of notation, g' is a multi-valued function whereas $g *_{\rm s} \lim$ is a strong equivalence class. So, more precisely, one could say $g' \in g *_{\rm s} \lim$. If g is a cylinder, then $g' \in g *_{\rm s} \lim \subseteq g * \lim$. For ease of notation we mix equivalence classes and representatives as above, whenever no confusion is expected. Together with Corollary 5.10 we obtain the following observation.

Corollary 5.17. *Let f be a multi-valued map on represented spaces. If f is a cylinder, then $f' \equiv_{\rm W} f' \times \lim \equiv_{\rm W} f * \lim$.*

It is interesting to mention that this characterization of the derivative has the consequence that choice on Baire space $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ is equivalent to its own derivative (see Theorem 9.16).

6. Super Strong Weihrauch Reducibility

In this section we briefly mention a side result that yields a counterpart of a result in classical computability theory. Namely it is known that $A \leq_{\rm T} B \iff A' \leq_1 B'$ for all $A, B \subseteq \mathbb{N}$ (see, for instance, Proposition V.1.6 in [Odi89]). Here A' denotes the Turing jump of A and $\leq_{\rm T}$ and $\leq_1$ denote Turing reducibility and one-to-one reducibility, respectively. In order to obtain a similar result we need to find the counterpart of one-to-one reducibility $\leq_1$ for our context. For this purpose we will use the next notion.

Definition 6.1 (Limit extensional computability)**.** A function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is called *computable in a limit extensional way* if F is computable and there is a computable $f :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $\lim \circ F = f \circ \lim$.

We note that in this situation F is a computable realizer of f with respect to the representation lim of $\mathbb{N}^\mathbb{N}$ on the input and output side. That is, F has to be extensional in the sense that it maps two sequences that converge to the same result to two sequences that also converge to the same result. In fact, F is computable in a limit extensional way if and only if it is computable and extensional in this sense. It is obvious that some functions such as the identity id are computable in a limit extensional way.

Definition 6.2 (Super strong Weihrauch reducibility)**.** Let f and g be multi-valued functions on represented spaces. Then we write $f \leq_{\rm ssW} g$ and we say that f is *super strongly Weihrauch reducible* to g, if there are computable $H, K :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that K is even computable in a limit extensional way and such that $HGK \vdash f$ for all $G \vdash g$.

A special case of super strong reducibility is that K falls away (i.e. is the identity), which means that the reduction can be achieved with H alone. It is obvious that $f \leq_{\rm ssW} g \Longrightarrow f \leq_{\rm sW} g$. Now we obtain the following characterization.

Proposition 6.3 (Derivatives and super strong reducibility)**.** *Let f and g be multi-valued functions on represented spaces. We obtain*

$$f \leq_{\rm sW} g \iff f' \leq_{\rm ssW} g'.$$

Proof. Let us assume $f \leq_{\mathrm{sW}} g$. We revisit the proof of Proposition 5.6 (2). The function K_0 obtained there is computable in a limit extensional way, hence (d) implies $f' \leq_{\mathrm{ssW}} g'$. For the other direction let now $f' \leq_{\mathrm{ssW}} g'$. Then there is a K_0 which is computable in a limit extensional way such that (d) holds. Hence there is a computable K such that $K \lim = \lim K_0$. By the Jump Realization Lemma 5.4 we obtain (c) and hence (b) and (a), which means that $f \leq_{\mathrm{sW}} g$. $\square$

We give an example that shows that super strong Weihrauch reducibility cannot be replaced by strong Weihrauch reducibility in this result. In particular, super strong Weihrauch reducibility is actually stronger than strong reducibility.

Example 6.4. Let $c : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, p \mapsto \widehat{0}$ be the constant zero function on Baire space and let $p \in \mathbb{N}^{\mathbb{N}}$ be limit computable, but not computable. By $c_p :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ we denote the restriction of c to $\{p\}$. Then we obtain $c \not\leq_{\mathrm{W}} c_p$, since c_p is not pointed (has no computable point in the domain). In particular, $c \not\leq_{\mathrm{sW}} c_p$. On the other hand, we claim $c' \equiv_{\mathrm{sW}} c \circ \lim \equiv_{\mathrm{sW}} c_p \circ \lim \equiv_{\mathrm{sW}} c_p'$. Here c_p' is a restriction of c' and hence clearly $c_p' \leq_{\mathrm{ssW}} c'$. Moreover, there is a computable q such that $\lim(q) = p$ and hence c_p' is pointed and $c' \equiv_{\mathrm{sW}} c \leq_{\mathrm{sW}} c_p'$. It follows from Proposition 6.3 that $c' \not\leq_{\mathrm{ssW}} c_p'$.

It is clear from this example that a non-pointed f can have a pointed derivative f', but a pointed f always has a pointed derivative f'. Following the pattern above, one can introduce a super^n strong Weihrauch reducibility that characterizes strong reducibility of the n–th derivative. We will not make any use of super strong reducibility in the following.

7. Derived Coproducts

In Proposition 5.7 we have proved that $f' \sqcup g' \leq_{\mathrm{sW}} (f \sqcup g)'$, but we did not prove the inverse reduction. We will see in Example 12.9 that the inverse reduction does not hold in general. However, we can define a variant $\sqcup'$ of the coproduct $\sqcup$ that has the property that $f' \sqcup' g' \equiv_{\mathrm{sW}} (f \sqcup g)'$ holds. We call $\sqcup'$ the *derived coproduct*. The difference to the ordinary coproduct is that the parameter that selects the function that is applied is replaced by a sequence that converges to such a parameter. In order to formalize this concept, we recall the definition of the coproduct representation. Let (X, δ_X) and (Y, δ_Y) be represented spaces, then the coproduct representation $\delta_X \sqcup \delta_Y$ of $X \sqcup Y = (\{0\} \times X) \cup (\{1\} \cup Y)$ is defined by $(\delta_X \sqcup \delta_Y)\langle 0, p\rangle := \delta_X(p)$ and $(\delta_X \sqcup \delta_Y)\langle 1, p\rangle := \delta_Y(p)$. Analogously, the representation δ_X^* of X^* is defined by $\delta_X^*\langle n, p\rangle := \delta_X^n(p)$. Now we can define the derived coproduct just by replacing this representation by a suitable substitute.

Definition 7.1 (Derived operations)**.** Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be multi-valued functions on represented spaces (X, δ_X) and (Z, δ_Z) and Y, W. Then we define the *derived coproduct* $f \sqcup' g :\subseteq X \sqcup Z \rightrightarrows Y \sqcup W$ to be the same function as $f \sqcup g$, but with a different representation $\delta_X \sqcup' \delta_Z$ of $X \sqcup Z$, defined by

$$(\delta_X \sqcup' \delta_Z)\langle p, q\rangle := (\delta_X \sqcup \delta_Z)\,\langle \lim_{n\to\infty} p(n), q\rangle$$

for all $p, q \in \mathbb{N}^{\mathbb{N}}$ such that p is eventually constant. Analogously, we define the *derived closure* $f^{\rightarrow} :\subseteq X^* \rightrightarrows Y^*$ to be the function $f^* :\subseteq X^* \rightrightarrows Y^*$, but with the representation $\delta^{\rightarrow}$ on the input side:

$$\delta^{\rightarrow}\langle p, q\rangle := \delta^*\langle \lim_{n\to\infty} p(n), q\rangle.$$

The intuition behind this concept is that like in case of $f \sqcup g$ the two functions f and g are both available and we can choose with a parameter n which one to use, however, we do not have to determine this parameter in a preprocessing step at the beginning of the computation, but we can change our mind about which of

f and g is to be used finitely many times during the computation. An analogous description holds true for $f^{\rightarrow}$. It is not too difficult to see that the derived closure is actually a closure operator, i.e. it satisfies $f \leq_{\mathrm{sW}} f^{\rightarrow}$, $f^{\rightarrow\rightarrow} \leq_{\mathrm{sW}} f^{\rightarrow}$ and $f \leq_{\mathrm{sW}} g$ implies $f^{\rightarrow} \leq_{\mathrm{sW}} g^{\rightarrow}$. It is also easy to see that $f \sqcup g \leq_{\mathrm{sW}} f \sqcup' g$ and $f^* \leq_{\mathrm{sW}} f^{\rightarrow}$.

Proposition 7.2. *Let f and g be multi-valued functions on represented spaces. Then we obtain:*

(1) $(f \sqcup g)' \equiv_{\mathrm{sW}} f' \sqcup' g'$,
(2) $f^{*\prime} \equiv_{\mathrm{sW}} f'^{\rightarrow}$.

Proof. For two represented spaces (X, δ_X) and (Y, δ_Y) we have $(\delta_X \sqcup \delta_Y)' \equiv \delta_X' \sqcup' \delta_Y'$ and $(\delta_X^*)' \equiv (\delta')^{\rightarrow}$. This implies the claim. □

Arno Pauly (personal communication) pointed out the following result, which is another indication that the derived closure operation is quite natural.

Example 7.3. $\mathsf{LPO}^{\rightarrow} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.

Arno Pauly [Pau09] has studied several further parallelization operations, one of which is similar to ours.

8. Higher Classes of Computable Functions

In this section we want to discuss variants of classes of limit computable functions, weakly computable functions and functions computable with finitely many mind changes on higher levels of the Borel hierarchy. In particular, we are interested in characterizing complete functions in the respective classes and in understanding the behavior of these functions under composition. We start with introducing a useful terminology for higher classes of limit computable functions.

Definition 8.1 (Limit computability)**.** Let $n \in \mathbb{N}$. We say that a multi-valued function f on represented spaces is $(n+1)$*–computable*, if $f \leq_{\mathrm{W}} \lim^{\circ n}$.

Here $g^{\circ n}$ denotes the n–fold composition of a map $g :\subseteq X \rightrightarrows X$, i.e. $g^{\circ 0} = \mathrm{id}_X$, $g^{\circ 1} = g$, $g^{\circ 2} = g \circ g$ etc. In particular, 1–computable is the same as computable and 2–computable is the same as limit computable. It is easy to see that $\lim^{\circ(n+1)} \equiv_{\mathrm{sW}} \lim^{(n)}$, where the right-hand side is the $n-th$ derivative of lim. The $(n+1)$–computable functions are also called *effectively* $\boldsymbol{\Sigma}^0_{n+1}$*–measurable* and the following fact about composition of limit computable functions is known (see [Bra05]).

Fact 8.2. *Let $n, k \in \mathbb{N}$ and let f and g be multi-valued functions on represented spaces such that $g \circ f$ exists. If f is $(n+1)$–computable and g is $(k+1)$–computable, then $g \circ f$ is $(n+k+1)$–computable.*

This can also be deduced inductively from Theorem 5.14.

In computability theory a point $p \in \mathbb{N}^{\mathbb{N}}$ is called *low*$_k$ for $k \in \mathbb{N}$ if $p^{(k)} \leq_{\mathrm{T}} \emptyset^{(k)}$, where $p^{(k)}$ denotes the k–th Turing jump of p (see [Soa87]). This concept can be relativized straightforwardly and we say that p is $(n+1)$*–low*$_k$ if $p^{(k)} \leq_{\mathrm{T}} \emptyset^{(n+k)}$.[5] We just write *low* instead of *low*$_1$. In [BdBP] we have shown how the pointwise concept of lowness can be treated uniformly using the map $\mathfrak{L}$. Here we generalize this idea to higher variants of lowness.

Definition 8.3 (Low map)**.** Let $n, k \in \mathbb{N}$. We define $\mathfrak{L}_{k,n} := (J^{-1})^{\circ k} \circ \lim^{\circ(n+k)}$. For short we write $\mathfrak{L}_k := \mathfrak{L}_{k,0}$.

[5] We note that relativizing an equivalent characterization of low$_k$ leads to a different notion (see Lemma 6.3.5 in [Nie09]).

We note that $\mathfrak{L}_0 = \mathrm{id}$ and the low_0 points are identified with the computable points. We also note that $\mathfrak{L}_1 = \mathfrak{L}$ and $\mathfrak{L}_{k,n} \equiv_{\mathrm{sW}} \mathfrak{L}_k^{(n)}$ by Lemma 5.2. The definition immediately implies the following result.

Lemma 8.4. *Let $n, k \in \mathbb{N}$. We obtain for each $p \in \mathbb{N}^{\mathbb{N}}$ that p is $(n+1)$–low_k if and only if there exists a computable $q \in \mathbb{N}^{\mathbb{N}}$ such that $\mathfrak{L}_{k,n}(q) = p$.*

Now we can extend the concept of lowness from points in Baire space to multi-valued functions on represented spaces using the low maps $\mathfrak{L}_{k,n}$.

Definition 8.5 (Classes of low functions)**.** Let $n, k \in \mathbb{N}$. We call a multi-valued function f on represented spaces $(n+1)$*–low$_k$*, if $f \leq_{\mathrm{sW}} \mathfrak{L}_{k,n}$.

Since the class of transparent functions is closed under composition, it follows that all the $\mathfrak{L}_{k,n}$ are transparent by Fact 5.5. We use this fact for the proof of the following theorem.

Theorem 8.6 (Low computability)**.** *Let f be a multi-valued function on represented spaces and let $n, k \in \mathbb{N}$. Then the following are equivalent:*

(1) f is $(n+1)$–low$_k$,
(2) $g \circ f$ is $(n+k+1)$–computable for any $(k+1)$–computable g of suitable type.

Proof. Let $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multi-valued function on represented spaces and let $n, k \in \mathbb{N}$. If f is $(n+1)$–low_k, then $f \leq_{\mathrm{sW}} \mathfrak{L}_{k,n}$ and there are computable functions H, K such that $H\mathfrak{L}_{k,n}K \vdash f$. Since $\mathfrak{L}_{k,n}$ is transparent there is a computable K_0 such that $H\mathfrak{L}_{k,n}K = \mathfrak{L}_{k,n}K_0$. Let now $g :\subseteq (Y, \delta_Y) \rightrightarrows (Z, \delta_Z)$ be a multi-valued function on represented spaces with $g \leq_{\mathrm{W}} \lim^{\circ k}$. Since $\lim^{\circ k}$ is a cylinder, there are computable H_1, K_1 such that $H_1 \lim^{\circ k} K_1 \vdash g$. By Fact 5.5 there is a computable H_0 such that $H_0 J^{\circ k} = H_1 \lim^{\circ k} K_1$. We obtain that $H_0 J^{\circ k} \mathfrak{L}_{n,k} K_0 \vdash g \circ f$. Since $H_0 J^{\circ k} \mathfrak{L}_{k,n} K_0 = H_0 \lim^{\circ(n+k)} K_0$, this implies $g \circ f \leq_{\mathrm{sW}} \lim^{\circ(n+k)}$.

Let us now assume that $g \circ f \leq_{\mathrm{W}} \lim^{\circ(n+k)}$ for any $g \leq_{\mathrm{W}} \lim^{\circ k}$ of suitable type. We consider the function $g := J^{\circ k} \circ \delta_Y^{-1} :\subseteq Y \rightrightarrows \mathbb{N}^{\mathbb{N}}$. Since δ_Y^{-1} is computable and $J^{\circ k} \leq_{\mathrm{W}} \lim^{\circ k}$, we obtain $g \circ f \leq_{\mathrm{W}} \lim^{\circ(n+k)}$ by assumption. Since $\lim^{\circ(n+k)}$ is a cylinder, there are computable H, K such that $H \lim^{\circ(n+k)} K \vdash g \circ f$. By Fact 5.5 there is a computable K_0 such that $H \lim^{\circ(n+k)} K = \lim^{\circ(n+k)} K_0$. This means $\lim^{\circ(n+k)} K_0(p) \in J^{\circ k} \delta_Y^{-1} f \delta_X(p)$ for all $p \in \mathrm{dom}(f\delta_X)$. Hence we obtain $\delta_Y (J^{-1})^{\circ k} \lim^{\circ(n+k)} K_0(p) \in f\delta_X(p)$, which means that $\mathfrak{L}_{k,n} K_0 \vdash f$ or, in other words $f \leq_{\mathrm{sW}} \mathfrak{L}_{k,n} \equiv_{\mathrm{sW}} \mathfrak{L}_k^{(n)}$ and f is $(n+1)$–low_k. $\square$

This characterization shows that the $(n+1)$–low_k functions form a very natural class of functions that exhibits some maximality behavior. We also formulate the special case for low functions.

Corollary 8.7 (Low functions)**.** *The class of low functions is exactly the class of multi-valued functions f on represented spaces such that $g \circ f$ is limit computable for any limit computable g of suitable type.*

This observation generalizes Proposition 8.16 in [BdBP], which provides already one inclusion of this characterization.

We can express Theorem 8.6 also in terms of compositional products. One should notice the similarity between this characterization of the $(n+1)$–low_k functions and the definition of $(n+1)$–low_k points.

Corollary 8.8. $\lim^{\circ(n+k)} \equiv_{\mathrm{sW}} \lim^{\circ k} *_{\mathrm{s}} \mathfrak{L}_{k,n}$ *for all $n, k \in \mathbb{N}$.*

Here the reduction $\leq_{\mathrm{sW}}$ follows by composition of $J^{\circ k}$ with $\mathfrak{L}_{k,n}$. Corollary 8.7 also implies the following result.

Proposition 8.9. *Let f be a multi-valued function on represented spaces and a cylinder. Then $f' \equiv_{\mathrm{sW}} f' *_{\mathrm{s}} \mathfrak{L}$.*

Proof. Let $M := \{f_0 \circ g_0 : f_0 \leq_{\mathrm{sW}} f'$ and $g_0 \leq_{\mathrm{sW}} \mathfrak{L}\}$. Then $f' *_{\mathrm{s}} \mathfrak{L}$ is a member of the strong degree $\sup(M)$. By Theorem 5.14 we know that $f_0 \leq_{\mathrm{sW}} f'$ is equivalent to $f_0 = f_1 \circ f_2$ for some $f_1 \leq_{\mathrm{sW}} f$ and $f_2 \leq_{\mathrm{sW}} \lim$ since f is a cylinder. By Corollary 8.7 $M = \{f_1 \circ g_1 : f_1 \leq_{\mathrm{sW}} f$ and $g_1 \leq_{\mathrm{sW}} \lim\}$ since the composition of a limit computable f_2 with a low g_0 gives exactly all limit computable g_1. Hence, $f *_{\mathrm{s}} \lim$ is also a member of the strong degree $\sup(M)$ and we obtain $f' *_{\mathrm{s}} \mathfrak{L} \equiv_{\mathrm{sW}} f *_{\mathrm{s}} \lim \equiv_{\mathrm{sW}} f'$ by Corollary 5.16. $\square$

We recall that a multi-valued function f on represented spaces is called *weakly computable*, if $f \leq_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{WKL}$. We now introduce weakly n–computable functions using compositions with limit computable functions.

Definition 8.10 (Weak computability)**.** Let $n \in \mathbb{N}$ and let f be a multi-valued function on represented spaces. Then we say that f is *weakly $(n+1)$–computable*, if there are multi-valued functions g, h on represented spaces such that g is weakly computable, h is $(n+1)$–computable and $f = g \circ h$. We call the weakly 2–computable functions also *weakly limit computable.*

Since $f = g \circ h$ for weakly computable g and computable h implies $f \leq_{\mathrm{W}} g$, it follows that weakly 1–computable is the same as weakly computable. With an inductive application of Theorem 5.14 we immediately get the following characterization using derivatives.

Corollary 8.11 (Weak computability)**.** *Let f be a multi-valued function on represented spaces and let $n \in \mathbb{N}$. Then the following are equivalent:*

1. *$f \leq_{\mathrm{W}} \mathsf{WKL}^{(n)}$,*
2. *f is weakly $(n+1)$–computable.*

Using the Uniform Low Basis Theorem (see Fact 3.9), Fact 3.4 and $\mathsf{C}_{\{0,1\}^{\mathbb{N}}} \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{R}}$ it follows that $\mathsf{WKL}^{(n)} \leq_{\mathrm{sW}} \mathfrak{L}^{(n)} \equiv_{\mathrm{sW}} \mathfrak{L}_{1,n}$. That is, we obtain the following corollary that shows that we have a hierarchy of concepts.

Corollary 8.12. *Let $n \in \mathbb{N}$ and let f be a multi-valued function on represented spaces. Then we obtain f $(n+1)$–computable $\Longrightarrow$ f weakly $(n+1)$–computable $\Longrightarrow$ f $(n+1)$–low $\Longrightarrow$ f $(n+2)$–computable.*

The implications in this corollary cannot be reversed in general. This is known for $n = 0$ (see [BdBP]) and will be proved later for $n = 1$ (see Theorem 12.7). We can derive some facts about the composition of classes of weakly computable functions.

Theorem 8.13 (Composition of weakly computable functions)**.** *Let $n, k \in \mathbb{N}$ and let f and g be multi-valued functions on represented spaces such that $g \circ f$ exists. Then we obtain the following:*

1. *If f is weakly $(n+1)$–computable and g is $(k+2)$–computable, then $g \circ f$ is $(n+k+2)$–computable.*
2. *If f is weakly $(n+1)$–computable and g is weakly $(k+1)$–computable, then $g \circ f$ is weakly $(n+k+1)$–computable.*

Proof. (1) In case $k = 0$ this follows directly from Theorem 8.6 since any weakly $(n+1)$–computable f is $(n+1)$–low by Corollary 8.12. In case $k \geq 1$ any $(k+2)$–computable g can be written as $g = g_0 \circ g_1$ with a $(k+1)$–computable g_0 and a 2–computable g_1 by Theorem 5.14. Hence the case $k \geq 1$ follows from the case $k = 0$ with the help of Fact 8.2.

(2) In case $n = k = 0$ this is well-known (see, for instance, Theorem 6.14 in [GM09], Proposition 7.11 in [BG11b] or Corollary 7.6 in [BdBP]) and this case implies the case for $k = 0$ and $n \in \mathbb{N}$; the statement (2) for $k \geq 1$ follows from (1). □

In particular, the weakly $(n+1)$–computable functions are closed under composition with weakly computable functions from right and left.

Another remarkable property of weakly $(n+1)$–computable functions is that they are automatically $(n+1)$–computable, if they are single-valued (under mild hypotheses on the target spaces).

Theorem 8.14 (Single-valuedness)**.** *Let X be a represented space and let Y be a computable metric space and let $n \in \mathbb{N}$. If $f :\subseteq X \to Y$ is weakly $(n+1)$–computable and single-valued, then f is $(n+1)$–computable.*

Proof. We prove the claim by induction on n. For $n = 0$ the claim has been proved in Corollary 8.8 of [BG11b].[6] Let f now be weakly $(n+2)$–computable. Then $f \leq_{\mathrm{sW}} \mathsf{WKL}^{(n+1)}$ by Corollary 8.11, since WKL is a cylinder. Then there is a represented space Z and $g :\subseteq Z \rightrightarrows Y$, $h :\subseteq X \rightrightarrows Z$ such that $g \leq_{\mathrm{sW}} \mathsf{WKL}^{(n)}$, $h \leq_{\mathrm{sW}} \lim$ and $f = g \circ h$ by Theorem 5.14 and since $\mathsf{WKL}^{(n)}$ and lim are cylinders. Since f is single-valued, it follows that the restriction $g_1 := g|_{\mathrm{range}(h)} :\subseteq Z \to Y$ of g to range(h) is single-valued too. Moreover, $g_1 \leq_{\mathrm{sW}} g \leq_{\mathrm{sW}} \mathsf{WKL}^{(n)}$. Hence, by Corollary 8.11 g_1 is weakly $(n+1)$–computable and by induction hypothesis we obtain that g_1 is $(n+1)$–computable. Hence $f = g_1 \circ h$ is $(n+2)$–computable. This completes the induction. □

Another important class of functions is the class of functions that are computable with finitely many mind changes. We recall that a multi-valued function f on represented spaces is called *computable with finitely many mind changes*, if $f \leq_{\mathrm{W}} \lim_\Delta$, where $\lim_\Delta$ is the limit operation on Baire space with respect to the discrete topology (see Theorem 7.11 in [BdBP]). The limit of a sequence in Baire space with respect to the discrete topology exists if and only if the sequence is eventually constant. This corresponds to a limit computation with finitely many mind changes on the output. We recall that $\lim_\Delta$ is a cylinder, see Fact 3.7. We generalize the class of functions computable with finitely many mind changes analogously to the class of weakly computable functions.

Definition 8.15 (Relativized computability with finitely many mind changes)**.** Let $n \in \mathbb{N}$ and let f be a multi-valued function on represented spaces. Then we say that f is $(n+1)$*–computable with finitely many mind changes*, if there are multi-valued functions g, h on represented spaces such that g is computable with finitely many mind changes, h is $(n+1)$–computable and $f = g \circ h$. We call the functions that are 2–computable with finitely many mind changes also *limit computable with finitely many mind changes*.

It is clear that the functions which are 1–computable with finitely many mind changes are just the functions which are computable with finitely many mind changes. With Theorem 5.14 we immediately get the following corollary.

Corollary 8.16 (Computability with finitely many mind changes)**.** *Let f be a multi-valued function on represented spaces and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) f is $(n+1)$–computable with finitely many mind changes,

[6] The result for $n = 0$ can be seen as a uniform version of the well-known fact that a unique infinite path in a computable binary tree is computable. However, the proof of the uniform version needs additional ideas, such as the application of a suitable topological selection theorem.

(2) $f \leq_{\mathrm{W}} \lim_\Delta^{(n)}$.

We point out that the derivatives $\lim_\Delta'$ and $\mathsf{C}_\mathbb{N}'$ are not Weihrauch equivalent, despite the fact that the underlying functions are (see Example 9.12). Hence we cannot replace $\lim_\Delta$ by $\mathsf{C}_\mathbb{N}$ in this corollary, except in the case $n = 0$. However, we can replace $\lim_\Delta$ by $\mathsf{C}_\mathbb{N} \times \mathrm{id}$ by Fact 3.7.

It is easy to see that the composition $g \circ f$ of a limit computable function g with a function f that is computable with finitely many mind changes is limit computable again. This can also be deduced from a consequence of the Uniform Low Basis Theorem (see Fact 3.9), since $\lim_\Delta^{(n)} \leq_{\mathrm{sW}} \mathsf{C}_\mathbb{R}^{(n)} \leq_{\mathrm{sW}} \mathfrak{L}^{(n)} \equiv_{\mathrm{sW}} \mathfrak{L}_{1,n}$. That is, we obtain the following corollary that shows that we have a hierarchy of computability concepts.

Corollary 8.17. *Let $n \in \mathbb{N}$ and let f be a multi-valued function on represented spaces. Then we obtain f $(n+1)$–computable $\Longrightarrow$ f $(n+1)$–computable with finitely many mind changes $\Longrightarrow$ f $(n+1)$–low $\Longrightarrow$ f $(n+2)$–computable.*

The implications in this corollary cannot be reversed in general. This is known for $n = 0$ (see [BG11a] and [BdBP]) and will be proved later for $n = 1$ (see Theorems 12.4 and 12.7 and Proposition 12.8).

Using Fact 8.2 and Theorem 8.6 we can derive some facts about the composition of classes of functions that are computable with finitely many mind changes.

Theorem 8.18 (Composition and mind changes)**.** *Let $n, k \in \mathbb{N}$ and let f and g be multi-valued functions on represented spaces such that $g \circ f$ exists. Then we obtain the following:*

(1) If f is $(n+1)$–computable with finitely many mind changes and g is $(k+2)$–computable, then $g \circ f$ is $(n+k+2)$–computable.

(2) If f is $(n+1)$–computable with finitely many mind changes and g is $(k+1)$–computable with finitely many mind changes, then $g \circ f$ is $(n+k+1)$–computable with finitely many mind changes.

This theorem can be proved analogously to Theorem 8.13 using Corollary 8.17 instead of Corollary 8.12 and using the fact that (2) is well-known in the case $n = k = 0$ by Corollary 7.6 in [BdBP]. In particular, the functions that are $(n+1)$–computable with finitely many mind changes are closed under composition with functions that are computable with finitely many mind changes from right and left.

We note that the class of functions bounded by $\mathsf{C}_\mathbb{R}$ is a common upper class of weakly computable functions and functions that are computable with finitely many mind changes by Example 4.4. This class is even smaller than the class of low functions and it is also closed under composition (by Theorem 8.7 and Corollary 7.6 in [BdBP]). We do not discuss generalizations of this class to higher levels here, although some straightforward conclusions follow from our results.

9. The Derivative of Closed Choice

In this section we want to characterize the derivative C_X' of closed choice C_X. We recall that a point $x \in X$ in a topological space X is called a *cluster point* of a sequence (x_n) in X, if each neighborhood U of x contains x_n for infinitely many $n \in \mathbb{N}$, that is $(\forall k)(\exists n \geq k)\, x_n \in U$. We mention that for metric spaces X a point x is a cluster point of a sequence (x_n) in X if and only if there is a subsequence of (x_n) that converges to x. This holds more generally for the larger class of Fréchet spaces (see Exercise 1.6.D in [Eng89]). We now study the cluster point map.

Definition 9.1 (Cluster point problem)**.** Let X be a computable metric space. We define

$$\text{Ł}_X : X^\mathbb{N} \to \mathcal{A}_-(X), (x_n) \mapsto \{x \in X : x \text{ is cluster point of } (x_n)\}.$$

We call $\mathsf{CL}_X := \mathsf{C}_X \circ \text{Ł}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X$ the *cluster point problem* of X.

We note that we consider Ł_X as a total map and hence we allow $\text{Ł}_X(x_n) = \emptyset$. However, we obtain $\text{dom}(\mathsf{CL}_X) = \{(x_n) : \text{Ł}_X(x_n) \neq \emptyset\}$. It is easy to see that the set of cluster points of a given sequence is always closed, hence the map Ł_X is actually well-defined. We immediately get an upper bound for Ł_X by showing that it is limit computable.

Proposition 9.2. *$\text{Ł}_X \leq_{\text{sW}} \lim$ for any computable metric space X.*

Proof. It is sufficient to show that Ł_X is limit computable. We use a computable standard enumeration (B_i) of the rational open balls of X. It follows from the definition of a cluster point that for all $x \in X$ the following holds:

$$x \notin \text{Ł}_X(x_n) \iff (\exists i)(x \in B_i \text{ and } (\exists k)(\forall n \geq k)\ x_n \notin B_i).$$

Moreover, for each $i \in \mathbb{N}$ the condition

$$(\exists k)(\forall n \geq k)\ x_n \notin B_i \tag{1}$$

implies $B_i \subseteq X \setminus \text{Ł}_X(x_n)$. Altogether it is sufficient to generate as output a list of all i that satisfy condition (1), since the union of the corresponding B_i is equal to $X \setminus \text{Ł}_X(x_n)$. There is clearly a limit machine that, given the sequence (x_n) can write the sequence $p \in \mathbb{N}^{\mathbb{N}}$ with

$$p\langle i, k\rangle := \begin{cases} 0 & \text{if } (\exists n \geq k)\ x_n \in B_i \\ 1 & \text{otherwise} \end{cases}$$

on its output tape in the limit. This is because the property $(\exists n \geq k)\ x_n \in B_i$ is c.e. open in all parameters. This limit machine can then be composed with an ordinary machine that enumerates all i on its output tape that satisfy the condition $(\exists k)\ p\langle i, k\rangle = 1$, which is equivalent to condition (1). Hence, the produced output constitutes a ψ_-–name of $\text{Ł}_X(x_n)$. □

Since $\mathsf{CL}_X = \mathsf{C}_X \circ \text{Ł}_X$, this proposition immediately implies $\mathsf{CL}_X \leq_{\text{sW}} \mathsf{C}_X'$ by Theorem 5.14. We will show that the inverse reduction holds as well. First we need a preliminary lemma about the existence of well-spaced nets in computable metric spaces.

Lemma 9.3. *For every computable metric space (X, d, α) there exists a computable function $h :\subseteq \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that:*

(1) $(\forall x \in X)(\forall s)(\exists n)\ d(\alpha(h(s, n)), x) < 2^{-s}$;
(2) for all s, n and $m < n$, if $(s, n) \in \text{dom}(h)$ then $(s, m) \in \text{dom}(h)$ and $d(\alpha(h(s, n)), \alpha(h(s, m))) > 2^{-s-1}$.

Proof. The definition is by recursion on n. For every s let $h(s, 0) = 0$. Assuming we have defined $h(s, 0), \ldots, h(s, n)$, for every $k > h(s, n)$ we check whether k satisfies one of the following c.e. tests:

(a) $(\exists m \leq n)\ d(\alpha(h(s, m)), \alpha(k)) < \frac{3}{4}2^{-s}$;
(b) $(\forall m \leq n)\ d(\alpha(h(s, m)), \alpha(k)) > 2^{-s-1}$.

Clearly for each k at least one of the tests succeeds, and we wait until one does. The least k for which (b) succeeds before (a) does is chosen as $h(s, n+1)$. If for all $k > h(s, n)$ test (a) succeeds before (b) does, then $h(s, n+1)$ is undefined and so are all $h(s, m)$ with $m > n$.

(2) is immediate from the definition.

To check (1) fix $x \in X$ and s. There exists k such that $d(\alpha(k), x) < \frac{1}{4}2^{-s}$. If $k = h(s, n)$ for some n we are done, otherwise let n be greatest such that $h(s, n) < k$ (n exists because $h(s, 0)$ is defined and $n \mapsto h(s, n)$ is strictly increasing). Since we did not set $h(s, n+1) = k$, test (a) succeeded with respect to k and n. Hence

there exists $m \le n$ such that $d(\alpha(h(s,m)),\alpha(k)) < \frac{3}{4}2^{-s}$. Then $d(\alpha(h(s,m)),x) < 2^{-s}$. □

We can now construct the desired reduction.

Theorem 9.4 (Derivative of Choice). $\mathsf{C}_X' \equiv_{\mathrm{sW}} \mathsf{CL}_X$ *for each computable metric space* X.

Proof. As mentioned before, $\mathsf{CL}_X \le_{\mathrm{sW}} \mathsf{C}_X'$ follows from Proposition 9.2 together with Theorem 5.14.

Given the computable metric space (X,d,α), to prove $\mathsf{C}_X' \le_{\mathrm{sW}} \mathsf{CL}_X$ fix h as in Lemma 9.3.

Let (p_n) be a sequence in $\mathbb{N}^{\mathbb{N}}$ such that $\lim_{n\to\infty} p_n = p$ and $\psi_-(p) = A \neq \emptyset$. We recall that ψ_- is a total representation and hence $p_n \in \mathrm{dom}(\psi_-)$ for all n. We want to find some element in A by using CL_X. We introduce the following notation. For every k and i with $p_k(i) = \langle j,l\rangle$, we let $c_i^k = \alpha(j)$ and $r_i^k = \overline{l}$, so that $B(c_i^k, r_i^k)$ is the i–th ball enumerated in $X \setminus \psi_-(p_k)$ by p_k. Similarly, let $B(c_i,r_i)$ be the i–th ball enumerated in $X \setminus A$ by p.

We define a sequence $H(p_n) \in X^{\mathbb{N}}$ by checking whether for each s and n the following c.e. test holds:

$$(\exists k \ge s)(\forall i \le s)\ d(c_i^k, \alpha(h(s,n))) > r_i^k - 2^{-s}.$$

Whenever we realize that some pair (s,n) passes the test, we put $\alpha(h(s,n))$ in the sequence we are defining. Notice that each (s,n) is responsible for enumerating $\alpha(h(s,n))$ in $H(p_n)$ at most once, although some point might occur repeatedly in $H(p_n)$ (because h is in general not one-to-one). The intuitive idea is that we want to approximate elements in A by points $\alpha(h(s,n))$ that tend to "escape" from the balls enumerated in $X \setminus A$ by p for $s \to \infty$. Our next claim implies that $H(p_n)$ is an infinite sequence belonging to the domain of CL_X.

We claim that every $x \in A \neq \emptyset$ is a cluster point of $H(p_n)$. Fix such an x, and recall that $d(c_i,x) \ge r_i$ for every i. For every s there exists n such that $d(\alpha(h(s,n)),x) < 2^{-s}$. We now show that $\alpha(h(s,n))$ occurs in $H(p_n)$. Since s is arbitrary, this shows that $x \in \mathsf{CL}_X H(p_n)$. Let $k \ge s$ be such that $c_i^k = c_i$ and $r_i^k = r_i$ for all $i \le s$. If $i \le s$ we have that

$$\begin{aligned} d(c_i^k, \alpha(h(s,n))) &\ge d(c_i,x) - d(x,\alpha(h(s,n))) \\ &> r_i - 2^{-s} = r_i^k - 2^{-s}. \end{aligned}$$

Thus $\alpha(h(s,n))$ occurs in $H(p_n)$.

To be sure that applying CL_X to $H(p_n)$ we obtain an element of A we need to check that no $x \in X \setminus A$ is a cluster point of the sequence. When $x \notin A$ we have $x \in B(c_i,r_i)$ for some i. There exists $m \ge i$ such that $c_i^k = c_i$ and $r_i^k = r_i$ for every $k \ge m$. Let $s_0 \ge m$ be such that $d(x,c_i) < r_i - 2^{-s_0}$ and set $\varepsilon = r_i - 2^{-s_0} - d(c_i,x) > 0$. If $s \ge s_0$ and $\alpha(h(s,n))$ appears in $H(p_n)$ because it satisfied the test with witness $k \ge s$ we have

$$\begin{aligned} d(\alpha(h(s,n)),x) &\ge d(\alpha(h(s,n)),c_i) - d(c_i,x) \\ &= d(\alpha(h(s,n)),c_i^k) - d(c_i^k,x) \\ &> r_i^k - 2^{-s} - d(c_i^k,x) \ge \varepsilon. \end{aligned}$$

Therefore, if $x \in \mathsf{CL}_X(H(p_n))$ then it is a cluster point of the elements of $H(p_n)$ of the form $\alpha(h(s,n))$ with $s < s_0$. This means that there exists a single $s_1 < s_0$ such that x is a cluster point of the elements of $H(p_n)$ of the form $\alpha(h(s_1,n))$. Since each (s_1,n) is responsible for enumerating $\alpha(h(s_1,n))$ in $H(p_n)$ at most once and $d(\alpha(h(s_1,n)),\alpha(h(s_1,m))) > 2^{-s_1-1}$ when $n \neq m$, this is clearly impossible. □

The proof, together with Proposition 9.2, actually yields the following stronger statement as well (we emphasize that there is a derivative ψ'_- on the output side).

Corollary 9.5. *Let (X,δ_X) be a computable metric space. Then the map*

$$Ł_X : (X^{\mathbb{N}},\delta_X^{\mathbb{N}}) \to (\mathcal{A}_-(X),\psi'_-)$$

as well as its multi-valued partial inverse $Ł_X^{-1}$ are computable.

This formulation has the benefit that it can be applied to certain restrictions of the cluster point problem and it immediately yields characterizations of their derivatives as well. We formulate an interesting characterization that can be derived from this result. We call a closed set $A \subseteq X$ *co-c.e. closed in the limit*, if $A = \psi'_-(p)$ for some computable p.

Corollary 9.6. *Let X be a computable metric space. Then a set $A \subseteq X$ is co-c.e. closed in the limit, if and only if it is the set of cluster points of some computable sequence (x_n) in (the dense subset of) X.*

The text in the parenthesis can be added (which can be deduced from the proof of Theorem 9.4) or omitted. The corollary generalizes a result of Le Roux and Ziegler (see Proposition 3.9 in [LRZ08]).

Now we continue to study special cases of the cluster point problem. We recall that by UCL_X we denote the cluster point problem restricted to sequences with a unique cluster point. Then $\mathsf{UCL}_X = \mathsf{UC}_X \circ Ł_X$, where UC_X denotes closed choice restricted to singletons. Again Proposition 9.2 (or the statement about $Ł_X$ in Corollary 9.5) together with Theorem 5.14 show that $\mathsf{UCL}_X \leq_{\mathrm{sW}} \mathsf{UC}'_X$. The inverse direction immediately follows from the statement on the inverse $Ł_X^{-1}$ in Corollary 9.5. We obtain the following corollary.

Corollary 9.7 (Derivative of unique closed choice)**.** $\lim_X \leq_{\mathrm{sW}} \mathsf{UC}'_X \equiv_{\mathrm{sW}} \mathsf{UCL}_X$ *for each computable metric spaces X.*

Here the first reduction holds since a converging sequence in a metric space has its limit as its unique cluster point. This result hence also provides a lower bound for the (unique) cluster point problem. An upper bound for the cluster point problem can be derived for many spaces from the following result. We recall that a computable metric space X is called a *computable K_σ–space*, if there exists a computable sequence (K_i) of non-empty computably compact sets $K_i \subseteq X$ such that $X = \bigcup_{i=0}^{\infty} K_i$ (see the discussion of computable compactness in Section 10 for further definitions). It was proved in Proposition 4.8 and Corollary 4.9 of [BdBP] that $\mathsf{C}_X \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{R}}$ for all computable K_σ-spaces. Since $\mathsf{C}_{\mathbb{R}}$ is a cylinder, this result is also true for strong reducibility. We combine this result with the Low Basis Theorem as stated in Fact 3.9. We recall that $\mathfrak{L} = J^{-1} \circ \lim$ and $\mathfrak{L}' \equiv_{\mathrm{sW}} J^{-1} \circ \lim'$.

Corollary 9.8 (Cluster point problem for K_σ–spaces)**.** $\mathsf{CL}_X \leq_{\mathrm{sW}} \mathsf{CL}_{\mathbb{R}} \leq_{\mathrm{sW}} \mathfrak{L}'$ *for all computable K_σ–spaces X.*

Here $\mathsf{CL}_X \leq_{\mathrm{sW}} \mathsf{CL}_{\mathbb{R}}$ follows from $\mathsf{C}_X \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{R}}$ by Theorem 9.4 and Proposition 5.6.

An immediate corollary of this result is the following. We say that a point $x \in X$ is *low relatively to the halting problem*, if it has a name $p \in \mathbb{N}^{\mathbb{N}}$ such that $p' \leq_{\mathrm{T}} \emptyset''$, i.e. if it is 2–low in the sense defined before.

Corollary 9.9. *Each computable sequence (x_n) of real numbers that has a cluster point at all, has a cluster point x that is low relatively to the halting problem.*

Obviously, this result holds true more generally for computable K_σ–spaces. If a metric space X is not K_σ in the classical sense, then one can embed Baire space $\mathbb{N}^{\mathbb{N}}$

into X and the cluster point problem becomes automatically much more difficult (see Theorem 9.16).

For the remainder of this section we discuss a number of examples of cluster point problems of certain spaces. We start with the cluster point problem on natural numbers, where we get the following immediate consequence of Proposition 3.8.

Corollary 9.10. $\mathsf{UCL}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{CL}_{\mathbb{N}} \equiv_{\mathrm{sW}} \lim'_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}'_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{UC}'_{\mathbb{N}}$.

Using Fact 3.7, Corollary 5.10 and the fact that $\lim_\Delta$ is strongly equivalent to the cylindrification of $\mathsf{C}_{\mathbb{N}}$, i.e. $\mathsf{C}_{\mathbb{N}} \times \mathrm{id} \equiv_{\mathrm{sW}} \lim_\Delta$ we obtain the following result.

Corollary 9.11. $\mathsf{UCL}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{UC}'_{\mathbb{R}} \equiv_{\mathrm{sW}} \lim'_\Delta \equiv_{\mathrm{sW}} \mathsf{CL}_{\mathbb{N}} \times \lim \equiv_{\mathrm{sW}} \mathsf{UCL}_{\mathbb{N}} \times \lim$.

Although $\mathsf{UC}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{R}}$, we point out that the respective derivatives are not equivalent (the equivalence between $\mathsf{UC}_{\mathbb{N}}$ and $\mathsf{UC}_{\mathbb{R}}$ is not a strong one). This is because $\mathsf{UCL}_{\mathbb{N}}$ maps computable inputs to computable outputs, whereas $\mathsf{UCL}_{\mathbb{N}} \times \lim$ does not. Hence we have another example for the fact that two strongly inequivalent members of the same Weihrauch degree can have inequivalent derivatives.

Example 9.12. $\mathsf{UC}'_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{UCL}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{UCL}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{UC}'_{\mathbb{R}}$ and $\mathsf{UC}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{R}}$.

Corollaries 9.11 and 8.16 together imply the following characterization of functions that are limit computable with finitely many mind changes, which states that the Unique Cluster Point Problem on the reals is complete for this class.

Corollary 9.13 (Limit computability with finitely many mind changes)**.** *Let f be a multi-valued function on represented spaces. Then the following are equivalent:*

(1) $f \leq_{\mathrm{W}} \mathsf{UCL}_{\mathbb{R}}$,
(2) f is limit computable with finitely many mind changes.

That leads to the following corollary, which is clear when x is a unique cluster point. If the cluster point is isolated, then one can easily identify those members of the sequence that are in some small isolating neighborhood of the point and hence one can reduce the case to the case of uniqueness. We note that any output written by a limit machine after finitely many mind changes is an ordinary limit computable point.

Corollary 9.14. *If x is an isolated cluster point of a computable sequence (x_n) of real numbers, then x is limit computable.*

Once again, this result can immediately be generalized to computable K_σ–spaces. For real numbers this was also proved by Le Roux and Ziegler (see Lemma 3.7 in [LRZ08]).

Now we study the (not necessarily unique) cluster point problem on reals.

Proposition 9.15. $\mathsf{CL}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{CL}_{\{0,1\}^{\mathbb{N}}} \times \mathsf{CL}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{CL}_{\{0,1\}^{\mathbb{N}}} \times \mathsf{UCL}_{\mathbb{R}}$.

Proof. It has been proved in Corollary 4.9 of [BdBP] that $\mathsf{C}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}} \times \mathsf{C}_{\mathbb{N}}$. This result can be strengthened to strong equivalence $\equiv_{\mathrm{sW}}$, since $\mathsf{C}_{\mathbb{R}}$ and $\mathsf{C}_{\{0,1\}^{\mathbb{N}}}$ are both cylinders, see Fact 3.2. Hence, with Proposition 5.7 and Theorem 9.4 we obtain

$$\mathsf{CL}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{C}'_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{C}'_{\{0,1\}^{\mathbb{N}}} \times \mathsf{C}'_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{CL}_{\{0,1\}^{\mathbb{N}}} \times \mathsf{CL}_{\mathbb{N}}.$$

Moreover, $\mathsf{CL}_{\{0,1\}^{\mathbb{N}}}$ is a cylinder and $\mathsf{CL}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{CL}_{\{0,1\}^{\mathbb{N}}} \times \lim$ by Corollary 5.10 and hence

$$\mathsf{CL}_{\{0,1\}^{\mathbb{N}}} \times \mathsf{CL}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{CL}_{\{0,1\}^{\mathbb{N}}} \times \lim \times \mathsf{CL}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{CL}_{\{0,1\}^{\mathbb{N}}} \times \mathsf{UCL}_{\mathbb{R}}$$

follows by Corollary 9.11. ∎

We note that despite the fact that $\mathsf{CL}_{\mathbb{N}}$ and $\mathsf{UCL}_{\mathbb{R}}$ are not equivalent, they can be exchanged here as a factor of $\mathsf{CL}_{\{0,1\}^{\mathbb{N}}}$, which is the derivative of a cylinder.

The following result characterizes the cluster point problem on Baire space. In this case the cluster point problem is exactly as difficult as closed choice on this space.

Theorem 9.16 (Cluster point problem on Baire space)**.** $\mathsf{CL}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{C}'_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$.

Proof. The equivalence $\mathsf{CL}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{C}'_{\mathbb{N}^{\mathbb{N}}}$ follows from Theorem 9.4. By the Independent Choice Theorem 7.3 and Corollary 7.5 in [BdBP] we obtain $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} * \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$. Since $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ is a cylinder and $\lim \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ (see Fact 3.2), it follows by Corollary 5.17 and Lemma 4.2 that

$$\mathsf{C}'_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} * \lim \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} * \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \leq_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}^{\mathbb{N}}}.$$

Since $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ is a cylinder, $\mathsf{C}'_{\mathbb{N}^{\mathbb{N}}}$ is a cylinder as well by Corollary 5.10. Hence the equivalence also holds for strong reducibility. □

So, in some sense, $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ behaves with respect to differentiability like the exponential function behaves with respect to analytic differentiability. We mention that this result has to be seen in light of the known fact that $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ is complete for single-valued (effectively) Borel measurable functions on computable metric space (see Fact 3.2).

We recall that $\mathsf{C}_{\mathbb{N}}, \mathsf{C}_{\{0,1\}^{\mathbb{N}}}, \mathsf{C}_{\mathbb{R}}$ and $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ are strongly idempotent and all these choice principles, except the first one, are also cylinders (see Fact 3.2). Hence $\mathsf{CL}_{\mathbb{N}}, \mathsf{CL}_{\{0,1\}^{\mathbb{N}}}, \mathsf{CL}_{\mathbb{R}}$ and $\mathsf{CL}_{\mathbb{N}^{\mathbb{N}}}$ have the respective properties by Corollaries 5.10 and 5.12.

Corollary 9.17. $\mathsf{CL}_{\{0,1\}^{\mathbb{N}}}, \mathsf{CL}_{\mathbb{R}}, \mathsf{CL}_{\mathbb{N}^{\mathbb{N}}}$ *are strongly idempotent and cylinders and* $\mathsf{CL}_{\mathbb{N}}$ *is strongly idempotent.*

Finally, we mention that the cluster point problem is always a strong fractal. The proof is basically the same as the proof of Proposition 5.8.

Corollary 9.18. CL_X *and* UCL_X *are strong fractals and hence strongly join-irreducible and join-irreducible for any computable space* X*.*

10. Compact Choice

In this section we want to consider the special case of the cluster point problem for sequences with relatively compact range. We recall that a set $A \subseteq X$ in a topological space X is called *relatively compact*, if its closure is compact. We will see that the following map is relevant in this context.

Definition 10.1 (Compact set of cluster points)**.** Let X be a computable metric space. We define

$$\mathsf{KL}_X :\subseteq X^{\mathbb{N}} \to \mathcal{K}_-(X), (x_n) \mapsto \{x \in X : x \text{ is cluster point of } (x_n)\},$$

with $\mathrm{dom}(\mathsf{KL}_X) := \{(x_n) : \overline{\{x_n : n \in \mathbb{N}\}} \text{ is compact}\}$.

Hence KL_X is a variant of the map L_X that we have studied before and it is easy to see that it is well-defined. There are two notable differences, for one we restrict KL_X to such input sequences that have a relatively compact range. Secondly, we require more output information, i.e. we want the set of cluster points with negative information as a compact set. The essential difference is that bounds need to be provided. We assume that $\mathcal{K}_-(X)$ is represented by κ_-, if not mentioned otherwise. Roughly speaking, a name p of a compact set $K = \kappa_-(p)$ is a list of all finite rational open covers $\mathcal{U} = \{B(x_1, r_1), ..., B(x_n, r_n)\}$ of K (see [BP03] for details). Here the x_i are supposed to be points in the dense subset and the r_i non-negative rational

numbers. We mention that the sets K with a computable κ_-–name are called *co-c.e. compact.* A *computably compact* set $K \subseteq X$ is one for which additionally all rational open balls that intersect K can be enumerated. A computable metric space X is called *computably compact*, if it is a co-c.e. compact subset of itself (which is equivalent to being a computably compact subset of itself in this special case).

In Proposition 9.2 we have seen that L_X is limit computable and in Corollary 10.7 we will prove the somewhat surprising fact that the same holds for KL_X. The fact that the input is given in positive form (as a sequence) enables us to compute the required additional output information in the limit at no extra costs, as Proposition 10.3 will show. For the proof we use some special version of the Lebesgue Covering Lemma, which is expressed formally in terms of the parameters of balls (see Theorem 4.3.31 in [Eng89] for the classical version).

Lemma 10.2 (Lebesgue Covering Lemma)**.** *Let X be some metric space and let $K \subseteq X$ be compact. Let (c_n) be a sequence in X and let (r_n) be a sequence of positive rational numbers. Then $K \subseteq \bigcup_{i\in\mathbb{N}} B(c_i, r_i)$ implies that there exists a $\varepsilon > 0$ such that for each $x \in K$ there is some $i \in \mathbb{N}$ with $d(x, c_i) < r_i - \varepsilon$.*

Proof. If $K \subseteq \bigcup_{i\in\mathbb{N}} B(c_i, r_i)$ then for each $x \in K$ there exists some $i_x = i \in \mathbb{N}$ with $x \in B(c_i, r_i)$ and some $\varepsilon_x > 0$ such that $d(x, c_i) < r_i - 2\varepsilon_x$. Now we have $K \subseteq \bigcup_{x\in K} B(x, \varepsilon_x)$ and since K is compact there is a finite subset $F \subseteq K$ such that $K \subseteq \bigcup_{y\in F} B(y, \varepsilon_y)$. We choose $\varepsilon := \min\{\varepsilon_y : y \in F\}$. Then for each $x \in K$ there is some $y \in F$ with $x \in B(y, \varepsilon_y)$ and for $i := i_y$ we have $d(y, c_i) < r_i - 2\varepsilon_y$ and hence $d(x, c_i) \leq d(x, y) + d(y, c_i) < \varepsilon_y + r_i - 2\varepsilon_y \leq r_i - \varepsilon$. □

The number ε is called a *Lebesgue covering number* of the respective cover $\{B(c_i, r_i) : i \in \mathbb{N}\}$. Now we are prepared to prove our main result.

Proposition 10.3 (Compact range)**.** *Let X be a computable metric space. The map*

$$R :\subseteq X^{\mathbb{N}} \to \mathcal{K}_-(X), (x_n) \mapsto \overline{\{x_n : n \in \mathbb{N}\}}$$

with $\mathrm{dom}(R) = \{(x_n) : \overline{\{x_n : n \in \mathbb{N}\}}$ *is compact*$\}$ *is limit computable, i.e.* $R \leq_{\mathrm{sW}} \lim$.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $K := \overline{\{x_n : n \in \mathbb{N}\}}$ is compact. For arbitrary points $c_1, ..., c_m$ in the dense subset of X and rational numbers $r_1, ..., r_m$ we claim that

$$K \subseteq \bigcup_{i=1}^{m} B(c_i, r_i) \iff (\exists k)(\forall n)(\exists i \in \{1, ..., m\})\, d(x_n, c_i) \leq r_i - 2^{-k}.$$

"$\Longrightarrow$" We assume $K \subseteq \bigcup_{i=1}^{m} B(c_i, r_i)$. Let $\varepsilon > 0$ be a Lebesgue covering number for this cover and let $k \in \mathbb{N}$ be such that $2^{-k} < \varepsilon$. Then the claim follows directly from the Lebesgue Covering Lemma 10.2.

"$\Longleftarrow$" Let $k \in \mathbb{N}$ be such that $(\forall n)(\exists i \in \{1, ..., m\})\, d(x_n, c_i) \leq r_i - 2^{-k}$. Let $x \in K$. We need to show that there is some $i \in \{1, ..., m\}$ with $x \in B(c_i, r_i)$. Since $x \in K$ there is some $n \in \mathbb{N}$ with $x_n \in B(x, 2^{-k})$. For this n there is some $i \in \{1, ..., m\}$ such that $d(x_n, c_i) \leq r_i - 2^{-k}$. Hence we obtain $d(x, c_i) \leq d(x, x_n) + d(x_n, c_i) < r_i$. This proves the claim.

We now describe a limit machine that lists all finite rational open covers $\mathcal{U}$ of K, given (x_n). In order to achieve this, we systematically test all possible covers $\mathcal{U} = \{B(c_i, r_i) : i \in \{1, ..., m\}\}$ together with all possible numbers k. We provisionally list $\mathcal{U}$ as a suitable cover on a specific position of the output and we try to verify the condition

$$(\exists n)(\forall i \in \{1, ..., m\})\, d(x_n, c_i) > r_i - 2^{-k}.$$

Since this condition is c.e. in all parameters, it can eventually be verified, if it is true. In this case the combination of $\mathcal{U}$ and k does not work and it will be replaced on the same output position by the cover $\mathcal{U}$ of the next combination of $\mathcal{U}$ and k. Eventually a combination for this position will be found that works and that will never be replaced. If, by dovetailing, this process is started countably many times for each output position in parallel with each possible combination of $\mathcal{U}$ and k as a starting combination of some output position, then in the end all suitable covers $\mathcal{U}$ are listed. □

We mention that the above algorithm computes a list of all finite open rational covers for the compact set K together with a corresponding Lebesgue covering number for each cover. However, we do not make any further use of the Lebesgue covering number on the output side. We note that Proposition 10.3 has also the following interesting corollary that we just mention as a side observation.

Corollary 10.4. *Let X be a computable complete metric space. Then the identity*

$$\mathrm{id} :\subseteq \mathcal{A}_+(X) \to \mathcal{K}_-(X), A \mapsto A,$$

restricted to compact sets as input is limit computable.

Here $\mathcal{A}_+(X)$ denotes the hyperspace of closed subsets with respect to positive information. In case of complete computable metric spaces, positive information on a set A can be given by a sequence (x_n) whose range is dense in A (see [BP03] for details).

We formulate a non-uniform corollary. We recall that a closed set $A \subseteq X$ is called *c.e. closed*, if there is a computable p with $\psi_+(p) = A$ and we call A *co-c.e. compact in the limit*, if $A = \kappa'_-(p)$ for some computable p.

Corollary 10.5. *Let X be a computable complete metric space. Any c.e. closed set $A \subseteq X$ that is also compact is co-c.e. compact in the limit.*

It is easy to see that the intersection of a closed set with a compact set is computable in the following sense (see Theorem 7.11 in [BG09] and the proof of Lemma 6 in [Bra08]).

Lemma 10.6 (Intersection)**.** *Let X be a computable metric space. Then the intersection operation $\cap : \mathcal{A}_-(X) \times \mathcal{K}_-(X) \to \mathcal{K}_-(X), (A, K) \mapsto A \cap K$ is computable.*

If we combine this result with Propositions 9.2 and 10.3 and the fact that lim is idempotent, then we obtain the following result.

Corollary 10.7. $\mathsf{KL}_X \leq_{\mathrm{sW}} \lim$ *for each computable metric space X.*

Now it is very natural to combine the function KL_X with compact choice K_X in the same way as we have combined L_X with closed choice C_X. Since slightly different versions of choice principles have been called "compact choice" in the past (see below) we define the one we need formally in order to be precise.

Definition 10.8 (Compact choice)**.** For each computable metric space X we call

$$\mathsf{K}_X :\subseteq \mathcal{K}_-(X) \rightrightarrows X, A \mapsto A$$

with $\mathrm{dom}(\mathsf{K}_X) := \{A \in \mathcal{K}_-(X) : A \neq \emptyset\}$ the *compact choice operation* of X.

In general the two variants of choice K_X and C_X are different from each other and also different from a third variant (also sometimes known as compact choice) denoted by KC_X, which is just C_X restricted to non-empty compact sets. In case of KC_X we only request information on these compact sets as closed sets, whereas in case of K_X, we request information on these sets as compact sets. We give an example to indicate that these principles are actually different. We recall that

by K_n and C_n we actually denote the respective choice operation K_X or C_X for $X = \{0, ..., n-1\}$.

Proposition 10.9. $\mathsf{K}_\mathbb{N} \equiv_{\mathrm{sW}} \mathsf{LLPO}^*$.

Proof. We claim that $\mathsf{K}_\mathbb{N} \equiv_{\mathrm{sW}} \bigsqcup_{n\in\mathbb{N}} \mathsf{C}_n \equiv_{\mathrm{sW}} \bigsqcup_{n\in\mathbb{N}} \mathsf{C}_2^n \equiv_{\mathrm{sW}} \mathsf{LLPO}^*$, where C_2^n denotes the n–fold product of C_2 with itself. Here the first reduction $\mathsf{K}_\mathbb{N} \leq_{\mathrm{sW}} \bigsqcup_{n\in\mathbb{N}} \mathsf{C}_n$ follows, since given a compact set $K \subseteq \mathbb{N}$ together with a bound $m \in \mathbb{N}$ such that $K \subseteq \{0, ..., m-1\}$, one can easily reduce this case to C_m. The reverse reduction is obvious. It follows from Theorem 31 in [Pau10a] (the proof even shows strong reducibility) that $\mathsf{C}_{n+1} \leq_{\mathrm{sW}} \mathsf{C}_2^n$ uniformly for all $n \in \mathbb{N}$. This implies $\bigsqcup_{n\in\mathbb{N}} \mathsf{C}_n \leq_{\mathrm{sW}} \bigsqcup_{n\in\mathbb{N}} \mathsf{C}_2^n$. The inverse reduction follows form Proposition 3.4 in [BdBP] (the proof even shows strong reducibility), which implies $\mathsf{C}_2^n \leq_{\mathrm{sW}} \mathsf{C}_{2^n}$ uniformly for all $n \in \mathbb{N}$. Moreover, $\mathsf{C}_2 \equiv_{\mathrm{sW}} \mathsf{LLPO}$ is easily proved, which implies $\bigsqcup_{n\in\mathbb{N}} \mathsf{C}_2^n \equiv_{\mathrm{sW}} \mathsf{LLPO}^*$. □

It follows from Proposition 3.8 that $\mathsf{C}_\mathbb{N} \equiv_{\mathrm{sW}} \mathsf{UC}_\mathbb{N} \equiv_{\mathrm{sW}} \mathsf{KC}_\mathbb{N}$, but it is known that $\mathsf{C}_\mathbb{N}$ is not reducible to LLPO^* (this follows from Lemma 4.1 in [BG11a]). Hence it is clear that in general K_X is different from C_X and KC_X. More precisely, we obtain the following corollary.

Corollary 10.10. $\mathsf{K}_\mathbb{N} <_{\mathrm{W}} \mathsf{KC}_\mathbb{N} \equiv_{\mathrm{sW}} \mathsf{C}_\mathbb{N}$.

This discrepancy between the different versions KC_X and K_X of compact choice disappears, however, for computably compact metric spaces X. This is because for such spaces, the identity $\mathrm{id} : \mathcal{A}_-(X) \to \mathcal{K}_-(X)$ is computable (see for instance Lemma 6 in [Bra08]).

Corollary 10.11. $\mathsf{K}_X \equiv_{\mathrm{sW}} \mathsf{KC}_X \equiv_{\mathrm{sW}} \mathsf{C}_X$ *for each computably compact computable metric space X.*

We mention two further facts that are known about K_X. For one, the following has been proved in Theorem 2.10 of [BG11a] (and essentially already in [GM09]).

Fact 10.12. $\mathsf{K}_X \leq_{\mathrm{sW}} \mathsf{K}_{\{0,1\}^\mathbb{N}}$ *for each computable metric space X.*

We recall that by a *computable embedding* $\iota : X \hookrightarrow Y$ we mean a computable injective map with a computable (partial) inverse. The following proof is essentially a simplified version of the proof of Proposition 4.3 in [BdBP].

Proposition 10.13. $\mathsf{K}_X \leq_{\mathrm{sW}} \mathsf{K}_Y$ *for all computable metric spaces X and Y with a computable embedding $\iota : X \hookrightarrow Y$.*

Proof. Let $\iota : X \hookrightarrow Y$ be a computable embedding. The map $J : \mathcal{K}_-(X) \to \mathcal{K}_-(Y), A \mapsto \iota(A)$ is known to be computable (see Theorem 3.3 in [Wei03]). One obtains $\mathsf{K}_X = \iota^{-1} \circ \mathsf{K}_Y \circ J$ and hence $\mathsf{K}_X \leq_{\mathrm{sW}} \mathsf{K}_Y$. □

Computable metric spaces X that admit a computable embedding $\iota : \{0,1\}^\mathbb{N} \hookrightarrow X$ have been called *rich* or *computably uncountable*. This class of spaces includes all computable Polish spaces X without isolated points (see Proposition 6.2 in [BG09]).

Corollary 10.14. $\mathsf{K}_X \equiv_{\mathrm{sW}} \mathsf{K}_{\{0,1\}^\mathbb{N}}$ *for all rich computable metric spaces X.*

11. The Bolzano-Weierstrass Theorem

The classical Bolzano-Weierstraß Theorem states that each bounded sequence (x_n) of real numbers has a cluster point x. We can easily generalize this statement to other spaces X and formulate our formal version BWT_X of the Bolzano-Weierstraß Theorem.

Definition 11.1 (Bolzano-Weierstraß Theorem)**.** Let X be a represented Hausdorff space. Then $\mathsf{BWT}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X$ is defined by

$$\mathsf{BWT}_X(x_n) := \{x \in X : x \text{ is a cluster point of } (x_n)\}$$

with $\mathrm{dom}(\mathsf{BWT}_X) := \{(x_n) \in X^{\mathbb{N}} : \overline{\{x_n : n \in \mathbb{N}\}} \text{ is compact}\}$.

Every sequence in a compact Hausdorff space has a cluster point (see Theorem 3.1.23 in [Eng89]), hence BWT_X is well-defined. We note that BWT_X is a total multi-valued function if X is a compact represented Hausdorff space. If X is a compact computable metric space, then there is no difference between the Bolzano-Weierstraß Theorem and the cluster point problem, i.e. we obtain $\mathsf{BWT}_X = \mathsf{CL}_X$.

The multi-valued function $\mathsf{BWT}_{\mathbb{R}}$ is the representative of the classical Bolzano-Weierstraß Theorem in the Weihrauch lattice and BWT_X can be considered as a generalization of the Bolzano-Weierstraß Theorem for arbitrary represented Hausdorff spaces X.

In the following we are interested in the case that X is a computable metric space and now we want to study the relation between compact choice K_X and the Bolzano-Weierstraß Theorem BWT_X. It is a straightforward observation that $\mathsf{BWT}_X = \mathsf{K}_X \circ \mathsf{KL}_X$. Since $\mathsf{KL}_X \leq_{\mathrm{sW}} \lim$ by Corollary 10.7, we immediately obtain $\mathsf{BWT}_X \leq_{\mathrm{sW}} \mathsf{K}_X'$ with Theorem 5.14. The inverse reduction then follows from the statement on the inverse L_X^{-1} in Corollary 9.5. The compact input information is not even required for this direction. Hence we obtain our following main result on the Bolzano-Weierstraß Theorem.

Theorem 11.2 (Bolzano-Weierstraß Theorem)**.** $\mathsf{BWT}_X \equiv_{\mathrm{sW}} \mathsf{K}_X'$ *for all computable metric spaces X.*

This theorem yields a good understanding of the Bolzano-Weierstraß Theorem and numerous consequences follow from this classification. This is mainly because we studied compact choice in detail and many properties can be transferred to the derivative.

For instance, Fact 10.12 has the following immediate corollary, which yields an upper bound for the Bolzano-Weierstraß Theorem on computable metric spaces.

Corollary 11.3. $\mathsf{BWT}_X \leq_{\mathrm{sW}} \mathsf{BWT}_{\{0,1\}^{\mathbb{N}}}$ *for each computable metric space X.*

Moreover, Proposition 10.13 implies the following result on embeddings.

Corollary 11.4. $\mathsf{BWT}_X \leq_{\mathrm{sW}} \mathsf{BWT}_Y$ *for all computable metric spaces X and Y with a computable embedding $\iota : X \hookrightarrow Y$.*

Corollary 10.14 can also be transferred to the Bolzano-Weierstraß Theorem.

Corollary 11.5. $\mathsf{BWT}_X \equiv_{\mathrm{sW}} \mathsf{BWT}_{\{0,1\}^{\mathbb{N}}}$ *for all rich computable metric spaces X.*

We mention a few concrete examples.

Corollary 11.6. $\mathsf{BWT}_{\mathbb{R}^n} \equiv_{\mathrm{sW}} \mathsf{BWT}_{\ell_2} \equiv_{\mathrm{sW}} \mathsf{BWT}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{BWT}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{N}^{\mathbb{N}}}$ *for all $n \geq 1$.*

The following corollary follows from Theorem 11.2 together with Fact 3.4 and Corollaries 10.11 and 11.6. It states that the Bolzano-Weierstraß Theorem on real numbers is nothing but the derivative of Weak Kőnig's Lemma.

Corollary 11.7 (Bolzano-Weierstraß as the derivative of Weak Kőnig's Lemma)**.** $\mathsf{WKL}' \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{R}}$.

It has been proved by Kleene that there are co-c.e. closed subsets $A \subseteq \{0,1\}^{\mathbb{N}}$ that have no computable point. One can choose, for instance, the set of all separating sets of a pair of computably inseparable c.e. sets (see Proposition V.5.25 in

[Odi89]). By a direct relativization of this construction one obtains that there is a set $A \subseteq \{0,1\}^{\mathbb{N}}$ that is co-c.e. closed in the limit and that has no limit computable point. Together with Corollary 11.7 and Fact 3.4 we obtain $\mathsf{C}'_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{R}}$ and hence the above example immediately yields the following result, which was also proved by Le Roux and Ziegler (see Theorem 3.6 in [LRZ08]).

Corollary 11.8. *There exists a computable bounded sequence (x_n) of real numbers that has no limit computable cluster point.*

This result holds more generally for sequences with relatively compact range in a rich computable metric space X because Corollaries 11.5 and 11.6 imply $\mathsf{C}'_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{BWT}_X$.

Corollary 11.9. *Let X be a rich computable metric space. Then there exists a computable sequence (x_n) in X with relatively compact range and without any limit computable cluster point.*

It turns out that Bolzano-Weierstraß for the natural numbers is the derivative of LLPO^*. This is a consequence of Theorem 11.2 and Proposition 10.9.

Corollary 11.10. $\mathsf{LLPO}^{*\prime} \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{N}}$.

Moreover, we obtain that Bolzano-Weierstraß for the two-point space $\{0,1\}$ is just the derivative of LLPO and this can be generalized to the finite case.

Corollary 11.11. $\mathsf{LLPO}' \equiv_{\mathrm{sW}} \mathsf{BWT}_2$ *and more generally for all $n \in \mathbb{N}$ we obtain* $\mathsf{C}'_n \equiv_{\mathrm{sW}} \mathsf{BWT}_n$.

We mention that $\mathsf{C}_n \equiv_{\mathrm{sW}} \mathsf{MLPO}_n$, where MLPO is a generalization of LLPO (see [BdBP]).

Finally, we can make the following observation on parallelization of the Bolzano-Weierstraß Theorem $\mathsf{BWT}_{\{0,1\}^{\mathbb{N}}}$, using Proposition 5.7 and Fact 3.4.

Corollary 11.12. $\mathsf{BWT}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \widehat{\mathsf{LLPO}}' \equiv_{\mathrm{sW}} \widehat{\mathsf{LLPO}'} \equiv_{\mathrm{sW}} \widehat{\mathsf{BWT}_2}$.

By Corollary 11.5 we know $\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{BWT}_{\{0,1\}^{\mathbb{N}}}$ and hence we get the following corollary.

Corollary 11.13. $\mathsf{BWT}_{\mathbb{R}}$ *is parallelizable, idempotent and a cylinder and* $\mathsf{BWT}_{\mathbb{N}}$ *is idempotent.*

Here parallelizability of $\mathsf{BWT}_{\mathbb{R}}$ follows from Corollaries 11.12 and 11.5 and it implies idempotency. Moreover, $\widehat{\mathsf{LLPO}} \equiv_{\mathrm{sW}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}}$ is known to be a cylinder by Facts 3.2 and 3.4 and so is its derivative by Corollary 5.10 and hence $\mathsf{BWT}_{\mathbb{R}}$ by Corollaries 11.12 and 11.5. Idempotency of $\mathsf{BWT}_{\mathbb{N}}$ follows from Corollary 11.10 since LLPO^* is strongly idempotent and hence its derivative by Corollary 5.12. Finally, we note that due to the fact that the cluster points of a sequence do not change if we extend the sequence by a finite prefix, we can conclude that the Bolzano-Weierstraß Theorem of any represented Hausdorff space is a strong fractal.

Proposition 11.14. BWT_X *and* UBWT_X *are strong fractals, join-irreducible and strongly join-irreducible for any represented Hausdorff space X.*

The proof is basically the same as the proof of Proposition 5.8 and in case of BWT_X for computable metric spaces X it follows immediately from Proposition 5.8 and Theorem 11.2. We obtain the following consequence of Corollary 9.8, which yields an upper bound for the Bolzano-Weierstraß Theorem.

Corollary 11.15 (Uniform Relative Low Basis Theorem)**.** $\mathsf{BWT}_{\mathbb{R}} \leq_{\mathrm{sW}} \mathsf{CL}_{\mathbb{R}} \leq_{\mathrm{sW}} \mathfrak{L}'$.

We immediately get the following non-uniform consequence.

Corollary 11.16. *Every bounded computable sequence (x_n) of real numbers has a cluster point x that is low relatively to the halting problem (i.e. x is 2–low).*

From Corollary 11.7 and Proposition 8.9 we can also derive the following observation (since WKL is a cylinder, see remark after Fact 3.4).

Corollary 11.17. $\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{R}} *_{\mathrm{s}} \mathfrak{L}$.

In other words, the functions below $\mathsf{BWT}_{\mathbb{R}}$ are stable under composition with low functions from the right. This allows us to strengthen Corollary 11.16 as follows.

Corollary 11.18. *Every bounded low sequence (x_n) of real numbers has a cluster point x that is low relatively to the halting problem.*

Another immediate consequence that we obtain here is that the Bolzano-Weierstraß Theorem is complete for the class of weakly limit computable functions. This follows from Corollaries 8.11 and 11.7.

Corollary 11.19. *Let f be a multi-valued function on represented spaces. Then the following are equivalent:*

(1) $f \leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$,
(2) f is weakly limit computable.

In particular, we obtain that typical single-valued functions below the Bolzano-Weierstraß Theorem are already limit computable.

Corollary 11.20. *Let X be a represented space, Y a computable metric space and let $f :\subseteq X \to Y$ be a single-valued function with $f \leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$. Then $f \leq_{\mathrm{W}} \lim$ follows, i.e. f is limit computable.*

Roughly speaking, this means that all problems reducible to the Bolzano-Weierstraß Theorem with a unique solution are limit computable. This is in particular applicable to the case of unique cluster points. By UBWT_X we denote the restriction of BWT_X to those sequences that have a unique cluster point. Then we obtain $\mathsf{UBWT}_{\mathbb{R}} \leq_{\mathrm{W}} \lim$. We will see that also the inverse reduction holds. We first prove a slightly more general result.

Proposition 11.21. $\lim_X = \mathsf{UBWT}_X$ *for each represented space X, which is a Hausdorff space.*

Proof. Let (x_n) be a sequence such that $K := \overline{\{x_n : n \in \mathbb{N}\}}$ is compact and let x be the unique cluster point of (x_n). Let U be an open neighborhood of x. Then $x_n \in U$ for infinitely many n. Let us assume that there are also infinitely many n with $x_n \notin U$. Then $x_n \in K \setminus U$ for infinitely many n and since $K \setminus U$ is a compact set, it follows that there is a cluster point $y \in K \setminus U$ of (x_n). In particular, $x \neq y$. This is a contradiction to the assumption that x is a unique cluster point of (x_n). Hence $x_n \in U$ for almost all n and hence x is the limit of (x_n).

If, on the other hand, (x_n) is a sequence that converges to some x and X is a Hausdorff space, then we claim that $K := \overline{\{x_n : n \in \mathbb{N}\}} = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ and K is compact. Here "$\supseteq$" follows since x is the limit of (x_n) and for "$\subseteq$" and compactness of K it suffices to show that $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact and hence closed in the Hausdorff space X by Theorem 3.1.8 in [Eng89]. Any open cover of $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ contains an open set U that contains x and hence almost all points x_n. This proves compactness of the set and finishes the proof of the claim. Clearly, x is a cluster point of (x_n). Let us assume that $y \in X$ is different from x. Since X is a Hausdorff space, x and y can be separated by open neighborhoods, i.e. there are open $U, V \subseteq X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Then $x_n \in U$ for almost all n and hence $x_n \in V$ for at most finitely many n. In particular, y cannot be a cluster point of (x_n) and x is the unique cluster point of this sequence. $\square$

This means, in particular, that $\lim_X \leq_{\mathrm{sW}} \mathsf{BWT}_X$ holds for all computable metric spaces X, which also gives us a lower bound on the complexity of the Bolzano-Weierstraß Theorem. Proposition 11.21 and Fact 3.5 yield the following result.

Corollary 11.22. $\mathsf{UBWT}_{\mathbb{R}} \equiv_{\mathrm{sW}} \lim$.

We mention the following immediate consequence, which is well-known and has a simple direct proof. Any computable convergent sequence without a computable limit is an example.

Corollary 11.23. *There is a computable sequence (x_n) of real numbers with a unique cluster point that is limit computable, but not computable.*

Another consequence of Proposition 11.21 is that the unique Bolzano-Weierstraß Theorem on $\mathbb{N}$ is just equivalent to choice on $\mathbb{N}$. This follows since $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \lim_{\mathbb{N}}$, see Proposition 3.8.

Corollary 11.24. $\mathsf{UBWT}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$.

The Bolzano-Weierstraß Theorem is often mentioned together with the Monotone Convergence Theorem, which says that a monotone growing bounded sequence of real numbers converges. We formalize this theorem in our lattice as well.

Definition 11.25 (Monotone Convergence Theorem)**.** The *Monotone Convergence Theorem* is the function

$$\mathsf{MCT} :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathbb{R}, (x_n) \mapsto \sup_{n\in\mathbb{N}} x_n$$

with $\mathrm{dom}(\mathsf{MCT}) = \{(x_n) : (\forall n)\ x_n \leq x_{n+1} \text{ and } (x_n) \text{ bounded}\}$.

In other words, MCT is just a restriction of the ordinary supremum function $\sup :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ (whose natural domain is just the set of all sequences that have a supremum) and it is easy to see that even $\mathsf{MCT} \equiv_{\mathrm{sW}} \sup$ holds. This is because any given sequence (x_n) that has a supremum can easily be converted into the sequence (y_n) with $y_n := \max\{x_0, ..., x_n\}$ that is monotone and has the same supremum. Hence we obtain the following observation (see for instance Proposition 3.7 in [BG11a]).

Fact 11.26. $\mathsf{MCT} \equiv_{\mathrm{sW}} \sup \equiv_{\mathrm{sW}} \lim$.

This allows us to formulate our main result about the Bolzano-Weierstraß Theorem as stated in Theorem 11.2 also in the following way: the Bolzano-Weierstraß Theorem is the compositional product of Weak Kőnig's Lemma and the Monotone Convergence Theorem. This follows from Corollary 5.16.

Corollary 11.27. $\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{WKL} * \mathsf{MCT}$.

12. Separation Results

In this section we want to discuss separation results that allow us to distinguish different versions of the Bolzano-Weierstraß Theorem and the cluster point problem from each other and from other degrees. One important separation technique already exploited in Theorem 4.4.2 of [BG11a] is the Computable Invariance Principle. On the one hand, this principle states that many notions of computability are preserved downwards by Weihrauch reducibility, for instance, if $f \leq_{\mathrm{W}} g$ and g is computable by a certain number of mind changes, then so is f. On the other hand, this principle also has a non-uniform variant, where Weihrauch degrees can be separated by considering Turing degrees of points. We formulate this principle in a slightly more general way here. We call a point x in a represented space *A–computable* for some $A \subseteq \mathbb{N}$ if x has a name $p \leq_{\mathrm{T}} A$. Analogously, we call x *A–low*

if x has a name p with $p' \leq_{\mathrm{T}} A'$. Here p' and A' denote the Turing jumps of p and A, respectively, and $A \oplus B := \{2n : n \in A\} \cup \{2n+1 : n \in B\}$ denotes the usual disjoint sum of $A, B \subseteq \mathbb{N}$.

Proposition 12.1 (Computable Invariance Principle). *Let f and g be multi-valued functions on represented spaces and let $A, B \subseteq \mathbb{N}$.*

1. *Let $f \leq_{\mathrm{W}} g$. If g has the property that for every A–computable $z \in \mathrm{dom}(g)$ there exists a B–computable $w \in g(z)$, then f has the property that for every A–computable $x \in \mathrm{dom}(f)$ there exists an $A \oplus B$–computable $y \in f(x)$.*
2. *Let $f \leq_{\mathrm{sW}} g$. If g has the property that for every A–computable $z \in \mathrm{dom}(g)$ there exists a B–computable $w \in g(z)$, then f has the property that for every A–computable $x \in \mathrm{dom}(f)$ there exists an B–computable $y \in f(x)$.*
3. *Let $f \leq_{\mathrm{sW}} g$. If g has the property that for every A–computable $z \in \mathrm{dom}(g)$ there exists a B–low $w \in g(z)$, then f has the property that for every A–computable $x \in \mathrm{dom}(f)$ there exists a B–low $y \in f(x)$.*

The third statement also holds true for $\leq_{\mathrm{W}}$ instead of $\leq_{\mathrm{sW}}$ when g is a cylinder or $A = \emptyset$.

Proof. (1) Let $f \leq_{\mathrm{W}} g$. Then there are computable functions $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, such that $H\langle \mathrm{id}, GK\rangle \vdash f$ whenever $G \vdash g$. Let g have the property that for every A–computable $z \in \mathrm{dom}(g)$ there is a B–computable $w \in g(z)$. Then by the Axiom of Choice there is some $G \vdash g$ with the property that $p \leq_{\mathrm{T}} A$ implies $G(p) \leq_{\mathrm{T}} B$ for all names p of any $z \in \mathrm{dom}(g)$. Hence, $q \leq_{\mathrm{T}} A$ implies $K(q) \leq_{\mathrm{T}} A$ and $GK(q) \leq_{\mathrm{T}} B$ for all names q of any $x \in \mathrm{dom}(f)$. We obtain $H\langle q, GK(q)\rangle \leq_{\mathrm{T}} \langle q, GK(q)\rangle \leq_{\mathrm{T}} A \oplus B$ for all names q of $x \in \mathrm{dom}(f)$. This means that f has the property that for every A–computable $x \in \mathrm{dom}(f)$ there is an $A \oplus B$–computable $y \in f(x)$.
(2) Can be proved analogously.
(3) Let now $f \leq_{\mathrm{sW}} g$. Then there are computable functions $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, such that $HGK \vdash f$ whenever $G \vdash g$. Let g have the property that for every A–computable $z \in \mathrm{dom}(g)$ there is a B–low $w \in g(z)$. Then by the Axiom of Choice there is some $G \vdash g$ with the property that $p \leq_{\mathrm{T}} A$ implies $(G(p))' \leq_{\mathrm{T}} B'$ for all names p of any $z \in \mathrm{dom}(g)$. Hence, $q \leq_{\mathrm{T}} A$ implies $K(q) \leq_{\mathrm{T}} A$ and $(GK(q))' \leq_{\mathrm{T}} B'$ for all names q of any $x \in \mathrm{dom}(f)$. We obtain $(HGK(q))' \leq_{\mathrm{T}} (GK(q))' \leq_{\mathrm{T}} B'$ for all names q of any $x \in \mathrm{dom}(f)$. This means that f has the property that for every A–computable $x \in \mathrm{dom}(f)$ there is a B–low $y \in f(x)$.
If g is a cylinder, then $f \leq_{\mathrm{W}} g$ implies $f \leq_{\mathrm{sW}} g$ and the extra claim follows from (3). If $A = \emptyset$, then $(H\langle q, GK(q)\rangle)' \leq_{\mathrm{T}} \langle q, GK(q)\rangle' \leq_{\mathrm{T}} (GK(q))' \leq_{\mathrm{T}} B'$ analogously to above for computable q. ☐

We note that we cannot strengthen the third result to ordinary Weihrauch reducibility, since $\langle q, GK(q)\rangle' \leq_{\mathrm{T}} \langle q', GK(q)'\rangle$ is not correct in general.

We now illustrate this proposition by generalizing the parallelization principle for higher derivatives that was provided in Lemma 4.1 of [BG11a]. This principle uses the closure properties of parallelization to separate degrees.

Theorem 12.2 (Higher parallelization principle). $\mathsf{LPO}^{(n)} \not\leq_{\mathrm{W}} \widehat{\mathsf{LLPO}}^{(n)}$ *for all $n \in \mathbb{N}$.*

Proof. Let us assume $\mathsf{LPO}^{(n)} \leq_{\mathrm{W}} \widehat{\mathsf{LLPO}}^{(n)}$. Then we obtain by parallelization

$$\lim^{(n)} \equiv_{\mathrm{W}} \widehat{\mathsf{LPO}}^{(n)} \leq_{\mathrm{W}} \widehat{\mathsf{LLPO}}^{(n)} \equiv_{\mathrm{W}} \mathsf{WKL}^{(n)}.$$

For this conclusion we have used Proposition 5.7, Facts 3.4 and 3.5 and the fact that the degrees mentioned here are all cylinders. We recall that $\mathsf{WKL}^{(n)} \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{R}}^{(n)} \leq_{\mathrm{sW}} \mathfrak{L}^{(n)}$

by the Uniform Low Basis Theorem 3.9. It follows that every tree that is $\emptyset^{(n)}$–computable has a path that is $\emptyset^{(n)}$–low. On the other hand, $\lim^{(n)}$ maps some inputs that are $\emptyset^{(n)}$–computable to outputs that are Turing equivalent to $\emptyset^{(n+1)}$ and hence not $\emptyset^{(n)}$–low. This is a contradiction to Proposition 12.1, because $\mathsf{WKL}^{(n)}$ is a cylinder. $\square$

Here we are mostly interested in the version of this result for $n = 1$, which we formulate as a corollary.

Corollary 12.3. $\mathsf{LPO}' \not\leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$.

We give an application of this principle, which shows that the Bolzano-Weierstraß Theorem on reals is incomparable with the unique cluster point problem on reals.

Theorem 12.4. $\mathsf{UCL}_{\mathbb{R}} \not\leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$ *and* $\mathsf{BWT}_{\mathbb{R}} \not\leq_{\mathrm{W}} \mathsf{UCL}_{\mathbb{R}}$*, as well as* $\mathsf{CL}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$ *and* $\mathsf{BWT}_{\mathbb{R}} \not\leq_{\mathrm{W}} \mathsf{CL}_{\mathbb{N}}$.

Proof. Since $\mathsf{CL}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{UCL}_{\mathbb{R}}$ by Corollary 9.11, it suffices to prove the second and third statement. The other two statements follow by transitivity. The second claim $\mathsf{BWT}_{\mathbb{R}} \not\leq_{\mathrm{W}} \mathsf{UCL}_{\mathbb{R}}$ follows from Corollary 11.8 together with Corollary 9.14. It is easy to see that $\mathsf{LPO} \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$ and hence we obtain with Corollary 9.10 and Proposition 5.6 that $\mathsf{LPO}' \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}' \equiv_{\mathrm{sW}} \mathsf{CL}_{\mathbb{N}}$. Hence the third claim $\mathsf{CL}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$ follows from Corollary 12.3. $\square$

Next we prove that BWT_2 is not limit computable.

Proposition 12.5. $\mathsf{BWT}_2 \not\leq_{\mathrm{W}} \lim$.

Proof. Let us assume there is a limit machine that computes BWT_2. Upon input of the constant zero sequence $p_0 = \widehat{0}$ the machine has to produce output 0 after only reading some prefix $w_0 \sqsubseteq p_0$. Upon input $p_1 := w_0\widehat{1}$ the limit machine will exhibit the same behaviour and eventually it has to change the output to 1 after reading only a prefix $w_1 \sqsubseteq p_1$. Continuing in this way one can construct a converging sequence (p_i) of the form $p_{2n+1} = w_{2n}\widehat{1}$ and $p_{2n+2} = w_{2n+1}\widehat{0}$ that converges to some p. Upon input of p the limit machine alternates the output for ever, which is not allowed for a limit machine. Hence, such a limit machine cannot exist. $\square$

As a preparation for the next result we prove that $\mathfrak{L}_{1,n}$ is not idempotent, which generalizes Theorem 8.8 in [BdBP].

Proposition 12.6. $\mathfrak{L}_{1,n}$ *is not idempotent for all* $n \in \mathbb{N}$.

Proof. Let $r \in \mathbb{N}^{\mathbb{N}}$ be such that $r \equiv_{\mathrm{T}} \emptyset^{(n+1)}$. By the Theorem of Spector (see Proposition V.2.26 in [Odi89]) there are $p, q \in \mathbb{N}^{\mathbb{N}}$ such that

$$\langle p, q\rangle \equiv_{\mathrm{T}} J(p) \equiv_{\mathrm{T}} J(q) \equiv_{\mathrm{T}} r.$$

Hence p, q are $(n+1)$–low, but $\langle p, q\rangle$ is not $(n+1)$–low. In particular, there are computable $s, t \in \mathbb{N}^{\mathbb{N}}$ such that $\lim^{\circ(n+1)}(s) = J(p)$ and $\lim^{\circ(n+1)}(t) = J(q)$. Hence $\langle \mathfrak{L}_{1,n} \times \mathfrak{L}_{1,n}\rangle\langle s, t\rangle = \langle p, q\rangle$ and the function $\mathfrak{L}_{1,n} \times \mathfrak{L}_{1,n}$ maps some computable inputs to values which are not $(n+1)$–low, in contrast to $\mathfrak{L}_{1,n}$, which maps all computable inputs to outputs that are $(n+1)$–low by Lemma 8.4. By Proposition 12.1 this means $\mathfrak{L}_{1,n} \times \mathfrak{L}_{1,n} \not\leq_{\mathrm{W}} \mathfrak{L}_{1,n}$. $\square$

We can now describe a strictly increasing finite chain of degrees related to the Bolzano-Weierstraß Theorem.

Theorem 12.7. $\mathsf{C}_{\mathbb{R}} <_{\mathrm{W}} \lim \equiv_{\mathrm{sW}} \mathsf{UBWT}_{\mathbb{R}} <_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}} <_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}} <_{\mathrm{W}} \mathfrak{L}' <_{\mathrm{W}} \lim' <_{\mathrm{W}} \mathsf{CL}_{\mathbb{N}^{\mathbb{N}}}$.

Proof. The strict reduction $\mathsf{C}_{\mathbb{R}} <_{\mathrm{W}} \lim$ was proved in Proposition 4.8 of [BG11a]. The equivalence $\lim \equiv_{\mathrm{sW}} \mathsf{UBWT}_{\mathbb{R}}$ was proved in Corollary 11.22. The reductions $\mathsf{UBWT}_{\mathbb{R}} \leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}} \leq_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}}$ are clear. Since $\mathsf{BWT}_2 \leq_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{R}}$, we clearly obtain $\mathsf{BWT}_{\mathbb{R}} \not\leq_{\mathrm{W}} \lim$ by transitivity and Proposition 12.5. Since $\mathsf{UCL}_{\mathbb{R}} \not\leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$ by Theorem 12.4 and $\mathsf{UCL}_{\mathbb{R}} \leq_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}}$, we obtain $\mathsf{CL}_{\mathbb{R}} \not\leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$ by transitivity. The reduction $\mathsf{CL}_{\mathbb{R}} \leq_{\mathrm{W}} \mathfrak{L}'$ was proved in Corollary 9.8 and since $\mathfrak{L} \leq_{\mathrm{sW}} \lim$, we obtain $\mathfrak{L}' \leq_{\mathrm{sW}} \lim'$. By Proposition 12.6 $\mathfrak{L}'$ is not idempotent, whereas $\lim'$ is clearly idempotent and $\mathsf{CL}_{\mathbb{R}}$ is idempotent by Corollary 9.17. Hence $\mathfrak{L}' \not\leq_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}}$ and $\lim' \not\leq_{\mathrm{W}} \mathfrak{L}'$ follow. By Theorem 9.16 we have $\mathsf{CL}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ and $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ is known to be complete for all single-valued function on computable metric spaces that are effectively Borel measurable, see Fact 3.2. In particular, we obtain $\lim' <_{\mathrm{W}} \lim'' \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$. □

Alternatively, lim can also be separated from $\mathsf{BWT}_{\mathbb{R}}$ using non-uniform results, such as Corollary 11.8 and Proposition 12.1(1). Analogously, $\lim'$ can be separated from $\mathfrak{L}'$ using Proposition 12.1(2). We mention that it follows from previous results that $\mathsf{BWT}_{\mathbb{R}} \sqcup \mathsf{UCL}_{\mathbb{R}}$ is strictly between $\mathsf{BWT}_{\mathbb{R}}$ and $\mathsf{CL}_{\mathbb{R}}$.

Proposition 12.8. $\mathsf{BWT}_{\mathbb{R}} <_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}} \sqcup \mathsf{UCL}_{\mathbb{R}} <_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}}$.

Proof. Since $\mathsf{BWT}_{\mathbb{R}} \sqcup \mathsf{UCL}_{\mathbb{R}}$ is the supremum of $\mathsf{BWT}_{\mathbb{R}}$ and $\mathsf{UCL}_{\mathbb{R}}$, which are incomparable by Theorem 12.4, it follows that $\mathsf{BWT}_{\mathbb{R}} <_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}} \sqcup \mathsf{UCL}_{\mathbb{R}}$. It is clear that $\mathsf{BWT}_{\mathbb{R}} \leq_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}}$ and $\mathsf{UCL}_{\mathbb{R}} \leq_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}}$ and we obtain $\mathsf{BWT}_{\mathbb{R}} \sqcup \mathsf{UCL}_{\mathbb{R}} \leq_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}}$. A supremum of two incomparable degrees is clearly not join-irreducible, but $\mathsf{CL}_{\mathbb{R}}$ is join irreducible by Corollary 9.18, hence $\mathsf{BWT}_{\mathbb{R}} \sqcup \mathsf{UCL}_{\mathbb{R}} <_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}}$ follows. □

Since $\mathsf{K}_{\mathbb{R}}' \sqcup \mathsf{UC}_{\mathbb{R}}' \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}} \sqcup \mathsf{UCL}_{\mathbb{R}}$ is not join-irreducible (as shown in the previous proof) and derivatives are join-irreducible by Proposition 5.8, we obtain the following example that shows that derivatives and coproducts do not commute.

Example 12.9. $\mathsf{K}_{\mathbb{R}}' \sqcup \mathsf{UC}_{\mathbb{R}}' <_{\mathrm{W}} (\mathsf{K}_{\mathbb{R}} \sqcup \mathsf{UC}_{\mathbb{R}})'$.

We now provide a strictly increasing finite chain of degrees related to the discrete Bolzano-Weierstraß Theorem.

Theorem 12.10. $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{UBWT}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{CL}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{UCL}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{UCL}_{\mathbb{R}} <_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}}$.

Proof. The first equivalence $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{UBWT}_{\mathbb{N}}$ has been proved in Corollary 11.24 and the equivalence $\mathsf{CL}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{UCL}_{\mathbb{N}}$ has been proved in Corollary 9.10. The reductions $\mathsf{UBWT}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{CL}_{\mathbb{N}}$ and $\mathsf{UCL}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{UCL}_{\mathbb{R}} \leq_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}}$ are obvious. We need to prove the strictness claims. By Proposition 12.5 we have $\mathsf{BWT}_2 \not\leq_{\mathrm{W}} \lim$. Since clearly $\mathsf{BWT}_2 \leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}}$ and $\mathsf{UBWT}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \lim$, we obtain $\mathsf{BWT}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{UBWT}_{\mathbb{N}}$ by transitivity. We mention that $\widehat{\mathsf{CL}_{\mathbb{N}}} \equiv_{\mathrm{W}} \widehat{\mathsf{C}_{\mathbb{N}}'} \equiv_{\mathrm{W}} \widehat{\mathsf{C}_{\mathbb{N}}}' \equiv_{\mathrm{W}} \lim'$, which follows from Corollary 9.10, Proposition 5.7, Fact 3.5 and Propositions 3.8 and 5.6. It is clear that we have $\mathsf{BWT}_2 \leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$ and we obtain by Corollaries 11.12, 11.13 and 11.6 that $\widehat{\mathsf{BWT}_{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$. Hence by Theorem 12.7 and parallelization it follows that $\widehat{\mathsf{BWT}_{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}} <_{\mathrm{W}} \lim' \equiv_{\mathrm{W}} \widehat{\mathsf{CL}_{\mathbb{N}}}$. Hence, $\mathsf{CL}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}}$. By Example 9.12 $\mathsf{UCL}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{UCL}_{\mathbb{R}}$. The strictness of the reduction $\mathsf{UCL}_{\mathbb{R}} <_{\mathrm{W}} \mathsf{CL}_{\mathbb{R}}$ follows from Proposition 12.8. □

13. Cardinality-Based Separation Techniques

In this section we discuss separation results for finite versions of the Bolzano-Weierstraß Theorem. For this purpose we will exploit that the Bolzano-Weierstraß Theorem BWT_X is obviously slim. We recall that for slim f we always have $\mathrm{range}(f) = \mathrm{range}(\mathsf{U}f)$. For slim functions we get the following necessary condition on the cardinality of ranges. The proof uses the Axiom of Choice. By $|X|$ we denote the *cardinality* of a set X.

Proposition 13.1. *Let f and g be multi-valued functions on represented spaces and let f be slim. Then $f \leq_{\mathrm{sW}} g \Longrightarrow |\mathrm{range}(f)| \leq |\mathrm{range}(g)|$.*

Proof. We consider multi-valued functions on represented spaces $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ and $g :\subseteq (W, \delta_W) \rightrightarrows (Z, \delta_Z)$. Then, by the Axiom of Choice there is some right inverse $S : Z \to \mathbb{N}^{\mathbb{N}}$ of δ_Z, i.e. $\delta_Z \circ S = \mathrm{id}_Z$. Let f be slim and $f \leq_{\mathrm{sW}} g$. Then there are computable H, K such that $HGK \vdash f$ for all $G \vdash g$. By the Axiom of Choice, there is some realizer $G \vdash g$. Without loss of generality we assume $\mathrm{dom}(G) = \mathrm{dom}(g\delta_W)$. Then $S\delta_Z G \vdash g$ follows and hence $HS\delta_Z GK \vdash f$. Since f is slim, for each $y \in \mathrm{range}(f)$ there is an $x \in \mathrm{dom}(f)$ such that $f(x) = \{y\}$. Let $p \in \mathbb{N}^{\mathbb{N}}$ be such that $\delta_X(p) = x$. Then $\delta_Y HS\delta_Z GK(p) \in f\delta_X(p) = \{y\}$. Hence

$$|\mathrm{range}(f)| \leq |\delta_Z GK(\mathrm{dom}(f\delta_X))| \leq |\mathrm{range}(g)|$$

follows. □

We would like to have a similar necessary criterion for ordinary Weihrauch reducibility. This criterion is harder to obtain since the direct access to the input gives a much higher degree of freedom and we will only be able to prove such a criterion in a special case. For this purpose we use strong fractals. As a side remark we mention that it follows from Proposition 13.1 that all slim strong fractals with target space $\mathbb{N}$ and at least two elements in the range are discontinuous.[7]

Lemma 13.2. *If $f :\subseteq X \rightrightarrows \mathbb{N}$ is a slim strong fractal and $|\mathrm{range}(f)| \geq 2$, then f is discontinuous.*

Proof. Let δ_X be the representation of X. If $f :\subseteq X \rightrightarrows \mathbb{N}$ is continuous then for each $p \in \mathrm{dom}(f\delta_X)$ there is some $w \sqsubseteq p$ such that $|\mathrm{range}(f_A)| = 1$ for $A = w\mathbb{N}^{\mathbb{N}}$. This implies $|\mathrm{range}(f_A)| = 1 < 2 = |\mathrm{range}(f)|$ and hence $f \not\leq_{\mathrm{sW}} f_A$ according to Proposition 13.1. This implies that f is not a strong fractal. □

Now we can formulate and prove a cardinality based separation principle for slim functions whose unique part is a strong fractal.

Theorem 13.3 (Cardinality condition for strong fractals)**.** *Let $f :\subseteq X \rightrightarrows \mathbb{N}$ and $g :\subseteq Z \rightrightarrows \mathbb{N}$ be multi-valued functions on represented spaces. If f is slim and $\mathsf{U}f$ is a strong fractal, then $f \leq_{\mathrm{W}} g \Longrightarrow |\mathrm{range}(f)| \leq |\mathrm{range}(g)|$.*

Proof. Let us assume that f is slim, $\mathsf{U}f$ is a strong fractal and $f \leq_{\mathrm{W}} g$. Let δ_X be the representation of X. Then there are computable H, K such that $H\langle \mathrm{id}, GK\rangle \vdash f$ for all $G \vdash g$. We assume that $\mathrm{dom}(K) = \mathrm{dom}(f\delta_X)$. For simplicity and without loss of generality we assume that G and H have target space $\mathbb{N}$. We consider the following claim: for each $i \in \mathbb{N}$ with $|\mathrm{range}(f)| > i$ there exists

(1) $k_i \in \mathrm{range}(f) \setminus \{k_0, ..., k_{i-1}\}$,
(2) $n_i \in \mathrm{range}(g) \setminus \{n_0, ..., n_{i-1}\}$,
(3) $p_i \in \mathrm{dom}(f\delta_X)$,
(4) $w_i \sqsubseteq p_i$,

such that $w_{i-1} \sqsubseteq w_i$, $GK(p_i) = n_i$ and $H\langle w_i\mathbb{N}^{\mathbb{N}}, n_i\rangle = k_i$. Let w_{-1} be the empty word. We prove this claim by induction on i. Firstly, if $|\mathrm{range}(f)| > 0$, then there exists some $k_0 \in \mathrm{range}(f)$ and since f is slim, there exists $p_0 \in \mathrm{dom}(f\delta_X)$ such that $f\delta_X(p_0) = \{k_0\}$. Let $n_0 := GK(p_0)$. By continuity of H there is some $w_0 \sqsubseteq p_0$ such that $H\langle w_0\mathbb{N}^{\mathbb{N}}, n_0\rangle = k_0$. Let $A_0 := w_0\mathbb{N}^{\mathbb{N}}$. Since A_0 is clopen and has non-empty intersection with $\mathrm{dom}(\mathsf{U}f\delta_X)$ and $\mathsf{U}f$ is a strong fractal, we obtain $\mathsf{U}f \leq_{\mathrm{sW}} \mathsf{U}f_{A_0}$. By Proposition 13.1 and since f is slim this implies $|\mathrm{range}(f)| = |\mathrm{range}(\mathsf{U}f)| \leq |\mathrm{range}(\mathsf{U}f_{A_0})|$. If $|\mathrm{range}(f)| > 1$, then there is some $k_1 \in \mathrm{range}(\mathsf{U}f_{A_0}) \setminus \{k_0\}$.

[7]We call a multi-valued function $f :\subseteq X \rightrightarrows \mathbb{N}$ *continuous*, if $f^{-1}\{n\} = \{x \in X : n \in f(x)\}$ is open in $\mathrm{dom}(f)$ for each $n \in \mathbb{N}$. Otherwise, f is called *discontinuous*.

Since $\mathsf{U}f_{A_0}$ is slim, there exists a $p_1 \in \mathrm{dom}(f\delta_X)$ such that $w_0 \sqsubseteq p_1$ and such that $f\delta_X(p_1) = \{k_1\}$. Let $n_1 := GK(p_1)$. Since $k_1 \neq k_0$, we obtain $n_1 \neq n_0$. By continuity of H there is some $w_1 \sqsubseteq p_1$ with $w_0 \sqsubseteq w_1$ such that $H\langle w_1\mathbb{N}^{\mathbb{N}}, n_1\rangle = k_1$. The proof can now continue inductively as above with $A_1 := w_1\mathbb{N}^{\mathbb{N}}$, which proves the claim. The claim implies $|\mathrm{range}(f)| \leq |\mathrm{range}(g)|$. $\square$

From this result we can derive a number of separation results for the Bolzano-Weierstraß Theorem of finite spaces. We recall that UBWT_X is always a strong fractal by Proposition 11.14 and BWT_X is obviously slim for any represented Hausdorff space X.

Theorem 13.4. $\mathsf{BWT}_n <_{\mathrm{W}} \mathsf{BWT}_{n+1} <_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$ *for all* $n \in \mathbb{N}$.

Proof. The reductions $\mathsf{BWT}_n \leq_{\mathrm{W}} \mathsf{BWT}_{n+1} \leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$ follow directly from Corollary 11.4. Since BWT_n is slim and UBWT_n is a strong fractal for all $n \in \mathbb{N}$, we obtain $\mathsf{BWT}_n <_{\mathrm{W}} \mathsf{BWT}_{n+1} <_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}}$ for all $n \in \mathbb{N}$ by Theorem 13.3. While $\mathsf{BWT}_{\mathbb{N}}$ always produces a computable output, the output of $\mathsf{BWT}_{\mathbb{R}}$ can even be necessarily not limit computable, see Corollary 11.8. This implies $\mathsf{BWT}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$ by Proposition 12.1. $\square$

Using Theorem 13.3 and similar arguments we can also prove the following result.

Theorem 13.5. $\lim_n <_{\mathrm{W}} \lim_{n+1} <_{\mathrm{W}} \lim_{\mathbb{N}} <_{\mathrm{W}} \lim_{\mathbb{R}}$ *for all* $n \in \mathbb{N}$.

Since $\lim_X = \mathsf{UBWT}_X$ for Hausdorff spaces X by Proposition 11.21, we get the following corollary for the unique version of Bolzano-Weierstraß Theorem.

Corollary 13.6. $\mathsf{UBWT}_n <_{\mathrm{W}} \mathsf{UBWT}_{n+1} <_{\mathrm{W}} \mathsf{UBWT}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{UBWT}_{\mathbb{R}}$ *for all* $n \in \mathbb{N}$.

We mention that all non-trivial (unique) versions of Bolzano-Weierstraß are above LPO.

Proposition 13.7. $\mathsf{LPO} <_{\mathrm{W}} \mathsf{UBWT}_2$.

Proof. $\mathsf{LPO} \leq_{\mathrm{W}} \lim_2$ is easy to see. Moreover, LPO can be computed with one mind change, whereas $\lim_2$ cannot be computed with any fixed number of mind changes.[8] This implies $\mathsf{LPO} <_{\mathrm{W}} \lim_2$ by the Mind Change Lemma 4.4 in [BG11a] and hence the claim since $\lim_2 = \mathsf{UBWT}_2$. $\square$

Since $\mathsf{LPO} \leq_{\mathrm{W}} \lim_n$ and $\mathsf{C}_n \equiv_{\mathrm{sW}} \mathsf{K}_n \leq_{\mathrm{sW}} \mathsf{K}_{\mathbb{R}} \equiv_{\mathrm{sW}} \widehat{\mathsf{LLPO}}$, we can use the parallelization principle as stated in Theorem 12.2 in order to conclude that $\lim_n \not\leq_{\mathrm{W}} \mathsf{C}_n$. The inverse reduction $\mathsf{C}_n \leq_{\mathrm{W}} \lim_n$ easily follows, since given a non-empty set $A \subseteq \{0, ..., n-1\}$ by negative information, one can always choose the smallest candidate of a member $x_i \in A$ that is not excluded by negative information at time step i in order to get a sequence (x_i) that converges to $x \in A$. Together with Proposition 12.5 we obtain the following corollary.

Corollary 13.8. $\mathsf{C}_n <_{\mathrm{W}} \mathsf{UBWT}_n = \lim_n <_{\mathrm{W}} \mathsf{BWT}_n \equiv_{\mathrm{sW}} \mathsf{C}_n'$ *for all* $n \geq 2$.

We mention that we also get the following consequence of Theorem 13.3.

Proposition 13.9. $\mathsf{UBWT}_{n+1} \not\leq_{\mathrm{W}} \mathsf{BWT}_n$ *for all* $n \in \mathbb{N}$.

It is also interesting to note that $\mathsf{UBWT}_n = \lim_n$ is complete for all limit computable functions with range of cardinality n.

Proposition 13.10. *Let* $f :\subseteq X \rightrightarrows \{0, ..., n-1\}$ *be a multi-valued function on represented spaces and* $n \in \mathbb{N}$. *Then the following are equivalent:*

[8]Intuitively, a multi-valued function is computable with at most n mind changes, if it can be computed by a Turing machine, which is allowed to revise its partial output at most n times altogether, see Definition 4.3 in [BG11a].

(1) $f \leq_{\mathrm{W}} \lim$*,*
(2) $f \leq_{\mathrm{sW}} \lim_n$*.*

An analogous results holds for $\mathbb{N}$ *instead of* $n = \{0, ..., n-1\}$*.*

Proof. It is clear that $f \leq_{\mathrm{sW}} \lim_n$ implies $f \leq_{\mathrm{W}} \lim$. Let us assume that $f \leq_{\mathrm{W}} \lim$. It is known that this means that there is a limit Turing machine that computes f. This means that the Turing machine has to stabilize the output on each output cell in the long run. Since for a discrete output only the first component of the output matters, we get a sequence of natural numbers in $\{0, ..., n-1\}$ that converges. This limit can be obtained with $\lim_n$. □

As a consequence we obtain the following result.

Theorem 13.11. $\lim_n = \mathsf{UBWT}_n \equiv_{\mathrm{sW}} \mathsf{BWT}_n \sqcap \lim$ *for all* $n \in \mathbb{N}$*.*

Proof. It is clear that $\mathsf{UBWT}_n \leq_{\mathrm{sW}} \mathsf{BWT}_n$ and $\lim_n \leq_{\mathrm{sW}} \lim$. Hence we obtain $\lim_n = \mathsf{UBWT}_n \leq_{\mathrm{sW}} \mathsf{BWT}_n \sqcap \lim$. We need to prove the inverse reduction. Let now $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multi-valued function on represented spaces, where Y is a computable Hausdorff space, i.e. a space Y such that the diagonal $\Delta_Y := \{(x, y) \in Y \times Y : x = y\}$ is co-c.e. closed. Let $f \leq_{\mathrm{sW}} \mathsf{BWT}_n$ and $f \leq_{\mathrm{sW}} \lim$. Then f is limit computable and there are computable H, K such that $HGK \vdash f$ for all $G \vdash \mathsf{BWT}_n$. Since BWT_n has target space $\{0, ..., n-1\}$, we can assume without loss of generality that this is also the target space of G and the source space of H. Let now $y_i := \delta_Y H(i)$ for $i = 0, ..., n-1$. Without loss of generality, we assume that all the y_i are pairwise different (otherwise we replace n by a suitable smaller n). Then the map $h : \{0, ..., n-1\} \to Y, i \mapsto y_i$ is clearly computable and bijective and since Y is a computable Hausdorff space and $\mathrm{dom}(h)$ is finite, the inverse h^{-1} is also computable. Hence $h^{-1}f :\subseteq X \rightrightarrows \{0, ..., n-1\}$ is limit computable and hence we obtain $f \equiv_{\mathrm{sW}} h^{-1}f \leq_{\mathrm{sW}} \lim_n$ by Proposition 13.10. This is, in particular, applicable to $f = \mathsf{BWT}_n \sqcap \lim$, since the output space $Y = (\{0\} \times \{0, ..., n-1\}) \cup (\{1\} \times \mathbb{N}^{\mathbb{N}})$ (with the coproduct representation) is a computable Hausdorff space. Hence we obtain $\mathsf{BWT}_n \sqcap \lim \leq_{\mathrm{sW}} \lim_n$. □

It is interesting to point out that there are compact computable metric spaces X such that the Bolzano-Weierstraß Theorem BWT_X is strictly between $\mathsf{BWT}_{\mathbb{N}}$ and $\mathsf{BWT}_{\mathbb{R}}$.

Proposition 13.12. $\mathsf{BWT}_X \not\leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}}$ *for all computable metric spaces* X *which are infinite and compact.*

Proof. We note that for compact X we have that $\mathrm{dom}(\mathsf{BWT}_X) = X^{\mathbb{N}}$ is compact. We use Schröder's computably admissible representation δ of $X^{\mathbb{N}}$, which is proper and hence $D = \mathrm{dom}(\delta) = \delta^{-1}(X^{\mathbb{N}})$ is compact (see [Wei03]). Moreover, we assume that the input space $\mathbb{N}^{\mathbb{N}}$ of $\mathsf{BWT}_{\mathbb{N}}$ is represented by the identity. Let us assume $\mathsf{BWT}_X \leq_{\mathrm{W}} \mathsf{BWT}_{\mathbb{N}}$. Then there are computable H, K such that $H\langle \mathrm{id}, GK\rangle \vdash \mathsf{BWT}_X$ for all $G \vdash \mathsf{BWT}_{\mathbb{N}}$. We note that $K(D) \subseteq \mathrm{dom}(\mathsf{BWT}_{\mathbb{N}}) = \{q \in \mathbb{N}^{\mathbb{N}} : (\exists m)(\forall k)\ q(k) \leq m\}$. We claim that there exist m and A non-empty and clopen in D such that

$$(\forall p \in A)(\forall k)\ K(p)(k) \leq m. \tag{2}$$

Since BWT_X is a strong fractal by Proposition 11.14 this claim and Corollary 11.4 imply

$$\mathsf{BWT}_{m+2} \leq_{\mathrm{sW}} \mathsf{BWT}_X \leq_{\mathrm{sW}} (\mathsf{BWT}_X)_A \leq_{\mathrm{sW}} \mathsf{BWT}_{m+1}$$

in contradiction to Theorem 13.4. We need to prove the existence of A and m that satisfy (2). Suppose there is no such suitable A and m. In particular, since $A = D$ and $m = 0$ do not satisfy (2), there exists a $p_0 \in D$ and k_0 such that $K(p_0)(k_0) > 0$.

Since K is continuous, there exists a clopen neighbourhood A_0 of p_0 in D such that $K(p)(k_0) > 0$ for all $p \in A_0$. At stage s we suppose that we have A_s, which is non-empty and clopen in D, $p_s \in A_s$ and k_s such that $K(p)(k_s) > s$ for all $p \in A_s$. Since $A = A_s$ and $m = s+1$ do not satisfy (2), there exists $p_{s+1} \in A_s$ and k_{s+1} such that $K(p_{s+1})(k_{s+1}) > s+1$. Using again the continuity of K we obtain $A_{s+1} \subseteq A_s$, which is clopen and non-empty in D such that $K(p)(k_{s+1}) > s+1$ for all $p \in A_{s+1}$. Since D is compact, there exists some $p \in \bigcap_{s=0}^{\infty} A_s$. Clearly, $K(p)(k_s) > s$ for all s in contradiction to $K(p) \in \mathrm{dom}(\mathsf{BWT}_{\mathbb{N}})$. This proves the claim. □

We give a concrete example.

Corollary 13.13. *Let* $X_{\omega+1} = \{-2^{-n} : n \in \mathbb{N}\} \cup \{0\}$ *with the Euclidean metric. Then* $\mathsf{BWT}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{BWT}_{X_{\omega+1}} <_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$.

Here the reductions to $\mathsf{BWT}_{\mathbb{R}}$ are strict since there are computable sequences whose (unique) cluster point is not computable, whereas $X_{\omega+1}$ only contains computable points.

14. Cluster Points versus Accumulation Points

We recall that a point x is called *accumulation point* of a subset $A \subseteq X$ of a topological space X, if each open neighbourhood U of x has a non-empty intersection with $A \setminus \{x\}$. By A' we denote the *set of accumulation points* of A (no confusion with the Turing jump is to be expected). We define the accumulation point problem as follows.

Definition 14.1 (Accumulation point problem)**.** Let X be a computable metric space. We consider the map

$$\mathsf{A}_X : \mathcal{A}_+(X) \to \mathcal{A}_-(X), A \mapsto A'.$$

We call $\mathsf{CA}_X := \mathsf{C}_X \circ \mathsf{A}_X$ the *accumulation point problem* of X.

We note that the input A is given with respect to positive information, whereas the output A' is required with negative information. The accumulation point problem is particularly well-behaved for these types of input and output information. In Theorem 9.6 of [BG09] we have proved that A_X is always limit computable. With Theorems 9.4 and 5.14 we immediately get the following corollary, which shows that the accumulation point problem is always reducible to the cluster point problem of the same space.

Corollary 14.2. $\mathsf{CA}_X \leq_{\mathrm{sW}} \mathsf{C}_X' \equiv_{\mathrm{sW}} \mathsf{CL}_X$ *for each computable metric space* X.

The inverse reduction cannot hold in general. For instance $\mathsf{CA}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_0$, since subsets of natural numbers have no accumulation points. However, the following result yields a substitute for the inverse result.

Proposition 14.3. $\mathsf{CL}_X \leq_{\mathrm{sW}} \mathsf{CA}_{X\times[0,1]}$ *for each computable metric space* X.

Proof. We note that for any sequence (x_n) in X, the set

$$A := \{(x_n, 2^{-n}) : n \in \mathbb{N}\} \subseteq X \times [0,1]$$

has the property that the set of its accumulation points is

$$A' = \{(x, 0) : x \text{ is cluster point of } (x_n)\}.$$

The map $X^{\mathbb{N}} \to \mathcal{A}_+(X \times [0,1])$ that maps any sequence (x_n) to the corresponding set A is computable. Likewise, the projection $\mathrm{pr} : X \times [0,1] \to X$ is computable. A combination of these operations yields the reduction. □

The space $[0,1]$ in this result could be replaced by $X_{\omega+1}$ from Corollary 13.13. For certain computable metric spaces X such as Euclidean space $\mathbb{R}$, Cantor space $\{0,1\}^{\mathbb{N}}$ and Baire space $\mathbb{N}^{\mathbb{N}}$ we know that $\mathsf{C}_{X\times[0,1]} \leq_{\mathrm{sW}} \mathsf{C}_X$ (see Section 7 of [BdBP]). The above results imply $\mathsf{CA}_{X\times[0,1]} \leq_{\mathrm{sW}} \mathsf{CL}_{X\times[0,1]} \leq_{\mathrm{sW}} \mathsf{CL}_X \leq_{\mathrm{sW}} \mathsf{CA}_{X\times[0,1]}$. Hence we get the following corollary.

Corollary 14.4. $\mathsf{CA}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{CL}_{\mathbb{R}}$, $\mathsf{CA}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{CL}_{\{0,1\}^{\mathbb{N}}}$, *and* $\mathsf{CA}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{CL}_{\mathbb{N}^{\mathbb{N}}}$.

15. The Contrapositive of the Bolzano-Weierstrass Theorem

In this section we consider the following contrapositive of the Bolzano-Weierstrass Theorem for sequences of reals: Every sequence in $\mathbb{R}$ eventually bounded away from each point of $[0,1]$ is eventually bounded away from the set $[0,1]$. Here, for any sequence (x_n) in $\mathbb{R}$,

- (x_n) *is eventually bounded away from* $x \in \mathbb{R}$ means that there exist $N \in \mathbb{N}$ and $\delta > 0$ such that $|x_n - x| > \delta$ for all $n \geq N$;
- (x_n) *is eventually bounded away from* $S \subseteq \mathbb{R}$ means that there exist $N \in \mathbb{N}$ and $\delta > 0$ such that $|x_n - x| > \delta$ for all $x \in S$ and $n \geq N$.

This statement is known in constructive mathematics (see e.g. [BB07, Bri09]) as the antithesis of Specker's Theorem. In particular in [BB07] it is proved that this principle is intuitionistically equivalent to a version of the Fan Theorem and therefore the authors consider it as an intuitionistic substitute for the Bolzano-Weierstraß Theorem. We use the following definition for the antithesis of Specker's Theorem.

Definition 15.1 (Antithesis of Specker's Theorem)**.** We call $\mathsf{AS} :\subseteq \mathbb{R}^{\mathbb{N}} \rightrightarrows \mathbb{N}\times\mathbb{N}$ with

$$\mathsf{AS}(x_n) := \{(N,k) \in \mathbb{N}\times\mathbb{N} : (\forall x \in [0,1])(\forall n \geq N)|x_n - x| > 2^{-k}\}$$

and $\mathrm{dom}(\mathsf{AS}) := \{(x_n) \in \mathbb{R}^{\mathbb{N}} : (\forall x \in [0,1])(\exists N,k)(\forall n \geq N)|x_n - x| > 2^{-k}\}$ the *antithesis of Specker's Theorem.*

The next proposition shows that in our setting the antithesis of Specker's Theorem is definitely simpler than the Bolzano-Weierstrass Theorem and in fact equivalent to the Baire Category Theorem by Fact 3.6.

Theorem 15.2 (Antithesis of Specker's Theorem)**.** $\mathsf{AS} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{BCT}$.

Proof. We first show that $\mathsf{AS} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$. For (x_n) eventually bounded away from each point of $[0,1]$ we let $B := \{\langle N,k\rangle \in \mathbb{N} : (\exists n \geq N)(-2^{-k} < x_n < 1+2^{-k})\}$. Then B is computably enumerable in (x_n), and with the help of $\mathsf{C}_{\mathbb{N}}$ we can obtain a point in $A = \mathbb{N}\setminus B$. For every $\langle N,k\rangle \in A$ we have that $d(x_n,x) > 2^{-(k+1)}$ for all $n \geq N$ and $x \in [0,1]$. Thus $(N,k+1) \in \mathsf{AS}(x_n)$.

We now show $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{AS}$. Let $p \in \mathbb{N}^{\mathbb{N}}$ be such that $\psi_-(p) = A$, i.e. $\mathbb{N}\setminus A = \{n \in \mathbb{N} : (\exists k)\ p(k) = n+1\}$. We now construct a sequence (x_n) that is eventually bounded away from each point of $[0,1]$ in the following way. Start with $x_0 = 2$. For every k we check whether $p(k) = x_{2k} - 1$. If this happens we let $x_{2k+1} := 0$ and $x_{2k+2} := \min\{i : (\forall n \leq k)\, p(n) \neq i+1\} + 2$, otherwise let $x_{2k+1} := x_{2k+2} := x_{2k}$. Since $A \neq \emptyset$, A has some least element, say $i \in \mathbb{N}$. Therefore (x_n) is eventually $i+2$ and hence is eventually bounded away from each point of $[0,1]$. Let now $(N,k) \in \mathsf{AS}(x_n)$. For every $n \geq N$, $x_n - 2 = i \in A$. $\square$

16. Conclusions

The diagram in Figure 1 illustrates some of the Weihrauch degrees that we have studied in this paper. The arrows indicate Weihrauch reducibility and not

necessarily strong reducibility. For details the reader should refer to the respective results.

We briefly compare our results with results that have been obtained in other approaches. We point out that not many exact transfer theorems between these different approaches are known, although obviously similar ideas emerge in different settings. More general comments in this direction can be found in [BG11a].

16.1. **Computable Analysis.** In computable analysis questions related to the Bolzano-Weierstraß Theorem have been studied in the past. For instance Mylatz [Myl92] has classified the complexity of the decision problem of whether a sequence contains a convergent subsequence. One obtains by the Theorem of Bolzano-Weierstraß that for a sequence (x_n) of real number the following holds:

$$(x_n) \text{ contains a cluster point } \iff (\exists i,j)(\forall k)(\exists n \geq k)\ x_n \in (i,j).$$

This shows that the set of sequences with cluster points is $\mathbf{\Sigma}^0_3$. It turns out that it is also $\mathbf{\Sigma}^0_3$–complete (see for instance Exercise 23.1 in [Kec95]) and hence the decision procedure is equivalent to $\mathsf{LPO}^{(2)}$. Moreover, von Stein [Ste89] has studied the decision problem of whether a given x is a cluster point of (x_n) and one easily sees that this can be phrased as

$$x \text{ is a cluster point of } (x_n) \iff (\forall i)(\forall k)(\exists n \geq k)\ d(x, x_n) < 2^{-i},$$

which is easily seen to be a $\mathbf{\Pi}^0_2$–complete property and hence this decision procedure is equivalent to LPO'. Le Roux and Ziegler [LRZ08] have studied, among other things, sets which are co-c.e. closed in the limit, they have provided a version of our Corollary 9.6 for Euclidean space and they have first proved that there exists a bounded computable sequence (x_n) of reals that has no limit computable cluster point.

16.2. **Constructive Analysis.** In constructive analysis Mandelkern has studied the Bolzano-Weierstraß Theorem. His main result is that the theorem is equivalent to LPO and the Monotone Convergence Theorem MCT (see [Man88, Ish04]). This can be understood from the perspective of our theory in light of the reduction $\mathsf{BWT}_\mathbb{R} \leq_{\mathrm{sW}} \widehat{\mathsf{LLPO}} *_{\mathrm{s}} \widehat{\mathsf{LPO}} \leq_{\mathrm{sW}} \widehat{\mathsf{LPO}'}$ and indeed Mandelkern proves the Bolzano-Weierstraß Theorem by a repeated and parallelized application of LPO. In the framework of constructive analysis one typically does not distinguish between parallelizations and compositional closures. The classification of $\mathsf{BWT}_\mathbb{R}$ being equivalent to LPO in the sense of constructive analysis is a very rough classification from our perspective and, in particular, it does not explain the computational differences between LPO, MCT and $\mathsf{BWT}_\mathbb{R}$. For instance, LPO always yields computable solutions, MCT always maps computable inputs to limit computable outputs, whereas $\mathsf{BWT}_\mathbb{R}$ maps some computable inputs necessarily to outputs that are not limit computable. On the other hand, our approach cannot distinguish certain constructive principles that are computably equivalent from our perspective. For example, principles such as LPO and WLPO (which is a weak version of LPO) are not intuitionistically equivalent, but equivalent in presence of Markov's principle. As Markov's principle is computable from our perspective, LPO and WLPO have equivalent Weihrauch degrees.

16.3. **Reverse Mathematics.** The situation in reverse mathematics is similar to the situation in constructive analysis. The Bolzano-Weierstraß Theorem $\mathsf{BWT}_\mathbb{R}$ is known to be equivalent to ACA_0 over RCA_0, see [Sim99]. The same holds true for the Monotone Convergence Theorem MCT. The system ACA_0 of arithmetic comprehension is the reverse mathematics counterpart of (the parallelization and compositional closure of) LPO (similarly as discussed above). That is, for a theorem

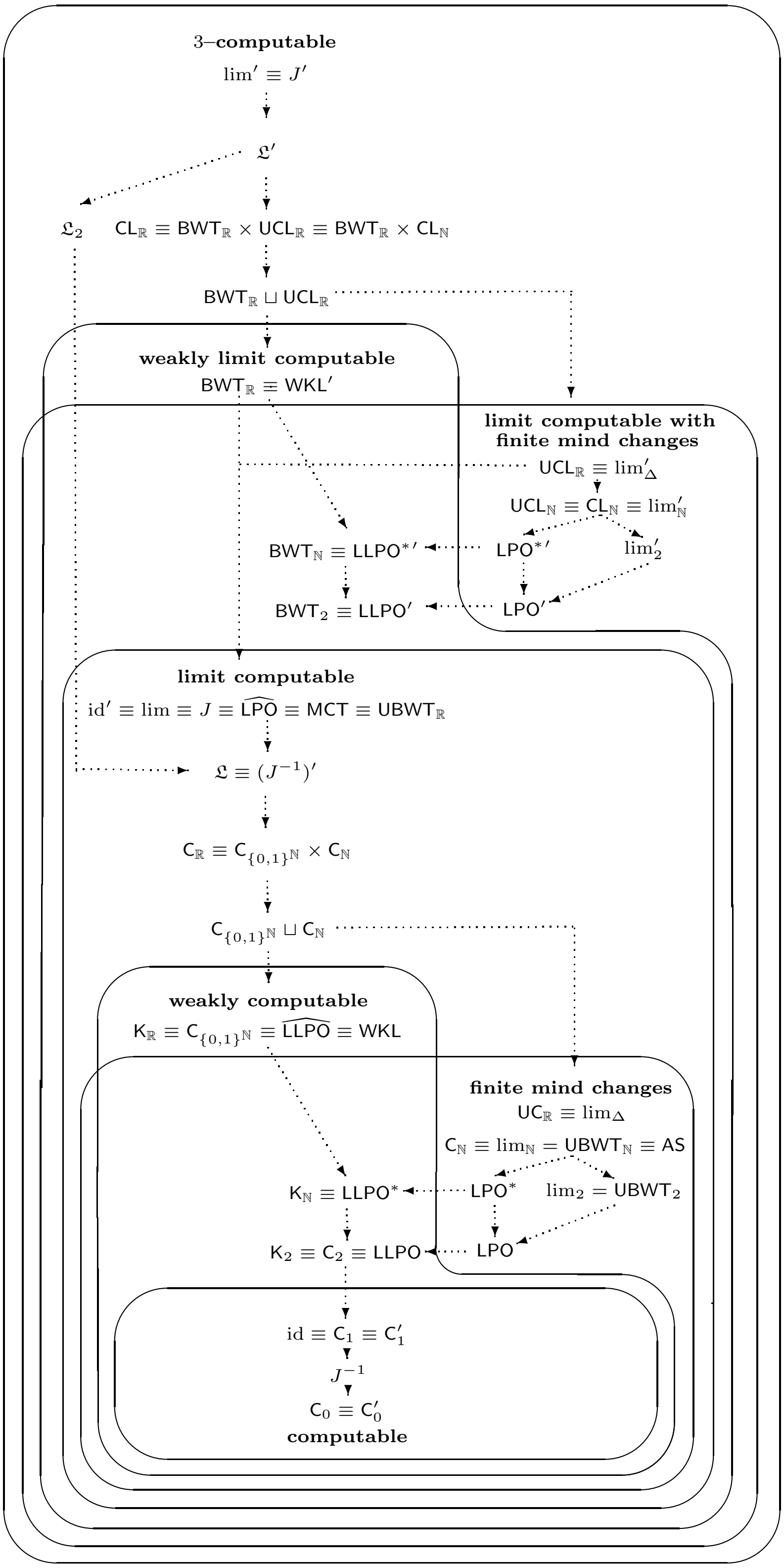

Figure 1. The Weihrauch Lattice

T being provable in ACA_0 roughly corresponds to the property that the analogous multi-valued function f (that formalizes T) satisfies $f \leq_{\rm W} \widehat{\mathsf{LPO}}^{(n)}$ for some $n \in \mathbb{N}$. The classification in constructive analysis is based on intuitionistic logic and hence uniform in our sense. In contrast to that, reverse mathematics is typically based on classical logic. Hence the classification rather corresponds to our non-uniform pointwise results.

16.4. **Proof Theory.** Reverse mathematics can be considered as a proof theoretic approach. However, there are also finer classifications of the Bolzano-Weierstraß Theorem in a proof theoretic setting (see Kohlenbach [Koh08] for a survey on this approach). Our jumps $\mathsf{LLPO}^{(n)}$ and $\mathsf{LPO}^{(n)}$ correspond to the proof theoretic principles $\mathbf{\Sigma}^0_{n+1}$-LLPO and $\mathbf{\Sigma}^0_{n+1}$-LEM, respectively, studied by Akama, Berardi, Hayashi and Kohlenbach [ABHK04]. Among many other things they proved that $\mathbf{\Sigma}^0_2$–LLPO does not imply $\mathbf{\Sigma}^0_2$–LEM, which can be seen as a counterpart of our Corollary 12.3. Our main results on the Bolzano-Weierstraß Theorem are closely related to results of Safarik and Kohlenbach, Kreuzer and perhaps even more closely to results of Toftdal. Toftdal [Tof04] has proved that the Bolzano-Weierstraß Theorem is instancewise equivalent to the principle $\mathbf{\Sigma}^0_2$-LLPO (over a weak intuitionistic base system). Kohlenbach, Safarik and Kreuzer have proved that instancewise the Bolzano-Weierstraß Theorem is equivalent to $\mathbf{\Sigma}^0_1-\mathsf{WKL}$ over RCA_0 (see [SK10, Kre11]). Here $\mathbf{\Sigma}^0_1-\mathsf{WKL}$ can be considered as the counterpart of our derivative WKL' of WKL. These results can be considered as analogous of our Corollary 11.7 in the respective settings. Our classification is fully uniform and does correspond rather to an even finer classification using linear logic. However, also some of the proof theoretic results mentioned above are already proved in a linear fashion. Exact metatheorems that allow translations from one setting to another one will have to be discussed elsewhere. We close with mentioning that very recently, Kreuzer has studied the Bolzano-Weierstraß Theorem for the Hilbert space ℓ_2, but with compactness interpreted in terms of the weak topology and this version of BWT turned out to be equivalent to $\lim''$, see [Kre].

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